

FANO VARIETIES WITH $\text{Nef}(X) = \text{Psef}(X)$ AND $\rho(X) = \dim X - 1$

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ABSTRACT. We classify mildly singular Fano varieties X such that $\text{Nef}(X) = \text{Psef}(X)$ and that the Picard number of X is equal to the dimension of X minus 1.

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1. INTRODUCTION

Let X be a complex normal projective variety. Let $\text{Nef}(X)$ be the cone of nef divisors in X and let $\text{Psef}(X)$ be the cone of pseudo-effective divisors. Then there is a natural inclusion

$$\text{Nef}(X) \subseteq \text{Psef}(X).$$

Consider the case when X is a Fano variety. If $\text{Nef}(X) = \text{Psef}(X)$ and X has log canonical singularities, then $\rho(X)$, the Picard number of X , is at most equal to the dimension of X (see [Dru14, Lem. 4.9]). If we assume further that $\rho(X) = \dim X$ and that X has factorial canonical singularities, then X is a product of double covers of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. More precisely, Druehl proves the follow theorem (see [Dru14, Thm. 1.1] and [Dru14, Prop. 10.4]).

Theorem 1.1. *Let X be a Fano variety with factorial canonical singularities such that $\rho(X) = \dim X$ and $\text{Nef}(X) \subseteq \text{Psef}(X)$. Then $X \cong X_1 \times \cdots \times X_k$ such that for all $i = 1, \dots, k$, either $X_i \cong \mathbb{P}^1$ or $\dim X_i \geq 3$ and X_i is a double cover of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, branched along a prime divisor of degree $(2, \dots, 2)$.*

In this paper, we consider Fano varieties X with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. If the dimension of X is 2, then X is isomorphic to \mathbb{P}^2 . In dimension 3, we prove the following theorem.

Theorem 1.2. *Let X be a Fano threefold with isolated locally factorial canonical singularities such that $\rho(X) = 2$ and $\text{Nef}(X) = \text{Psef}(X)$. Then one of the following holds.*

1. $X \cong \mathbb{P}^1 \times \mathbb{P}^2$.
2. X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$, branched along a prime divisor of degree $(2, 2)$.
3. X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$, branched along a prime divisor of degree $(2, 4)$.
4. X is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1, 1)$.

5. X is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1, 2)$.
6. X is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(2, 2)$.
7. X is a double cover of a smooth hypersurface Y in $\mathbb{P}^2 \times \mathbb{P}^2$ of degree $(1, 1)$, branched along a prime divisor D , which is the intersection of Y and a hypersurface of degree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

Note that if we assume that X is smooth, then Theorem 1.2 follows from the classification of smooth Fano threefolds due to Mori and Mukai ([MM82]).

In higher dimensions, we will first introduce some examples. In the following examples, we provide methods of constructions of Fano varieties X with $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$ (see Constriction 3.7 for more details).

Example 1.3. We will give two examples of Fano varieties X of dimension $n \geq 4$ which are finite covers of the product $Z = (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$, of degree 2 or 4.

1. Let X be a double cover over Z , branched along a divisor of degree $(2, \dots, 2, k)$, where $k \in \{2, 4\}$.
2. We write $Z = (\mathbb{P}^1)^s \times \mathbb{P}^2 \times (\mathbb{P}^1)^r$ such that $r, s > 0$ and $r + s = n - 2$. Let Y_1 be a double cover of $(\mathbb{P}^1)^s \times \mathbb{P}^2$, branched along a divisor of degree $(2, \dots, 2, 2)$. Let $Y = Y_1 \times (\mathbb{P}^1)^r$ and let $g : Y \rightarrow \mathbb{P}^2 \times (\mathbb{P}^1)^r$ be the natural projection. Let X be a double cover of Y , branched along a divisor D which is the pullback by g of some divisor of degree $(2, 2, \dots, 2)$ in $\mathbb{P}^2 \times (\mathbb{P}^1)^r$.

Example 1.4. Let $B_1 \cong B_2 \cong \mathbb{P}^2$. We will give three examples of Fano varieties X of dimension $n \geq 4$ which are finite covers of the product $Z = (\mathbb{P}^1)^{n-3} \times W$, of degree 2 or 4, where W is some normal ample hypersurface in $B_1 \times B_2$.

1. Let W be a smooth divisor of degree $(1, 1)$ in $B_1 \times B_2$. Let X be a double cover of $(\mathbb{P}^1)^{n-3} \times W$, branched along a divisor D , which is the intersection of $(\mathbb{P}^1)^{n-3} \times W$ and some divisor of degree $(2, \dots, 2, 2, 2)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$.
2. Let Y_1 be a double cover of $(\mathbb{P}^1)^{n-3} \times B_1$, branched along a divisor of degree $(2, \dots, 2, 2)$. Let $Y = Y_1 \times B_2$ and let p be the natural projection from Y to $B_1 \times B_2$. Let W be a normal hypersurface of degree $(1, k)$ in $B_1 \times B_2$, where $k \in \{1, 2\}$. Let $X = p^*W$.
3. Assume that $n \geq 5$. Let $V_1 = (\mathbb{P}^1)^r \times B_1$ and let $V_2 = B_2 \times (\mathbb{P}^1)^s$, where $r, s > 0$ and $r + s = n - 3$. Let Y_1 be a double cover of V_1 , branched along a divisor of degree $(2, \dots, 2, 2)$ and let Y_2 be a double cover of V_2 , branched along a divisor of degree $(2, 2, \dots, 2)$. Let $Y = Y_1 \times Y_2$. Let p be the projection from Y to $B_1 \times B_2$. Let W be a normal hypersurface of degree $(1, 1)$ in $B_1 \times B_2$. Let $X = p^*W$.

The main objective of this paper is to prove the following theorem.

Theorem 1.5. *Let X be a Fano variety of dimension at least 3, with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. Then $X = X_1 \times X_2$ such that*

- X_1 is either a point or one of the varieties in Theorem 1.1;
- X_2 is one of the varieties in Theorem 1.2, Example 1.3 and Example 1.4.

Note that this theorem is false without assuming the variety is smooth in codimension 2. In fact, even in dimension 2, there are surfaces other than \mathbb{P}^2 which have Picard number 1 and locally factorial canonical singularities. They are Fano surfaces which have exactly one singular point which is canonical of type E_8 (see [MZ88, Lem. 6]).

Outline of the proof of the theorems. Let X be a variety satisfying the condition in Theorem 1.5. We first consider the case when $\dim X = 3$. In this case, the Mori cone $\overline{\text{NE}}(X)$ has exactly two extremal rays R_1, R_2 . Let $f_i : X \rightarrow B_i$ be the extremal contraction with respect to R_i for $i = 1, 2$. We can prove that B_i is either \mathbb{P}^1 or \mathbb{P}^2 and the product $f_1 \times f_2$ is finite onto its image. Hence X is a finite cover of $\mathbb{P}^1 \times \mathbb{P}^2$ or a finite cover of a hypersurface of $\mathbb{P}^2 \times \mathbb{P}^2$. With the help of two results on finite morphisms between Fano threefolds, we can deduce the proof of Theorem 1.2.

After this, we consider the case when $n = \dim X \geq 4$. The proof of Theorem 1.5 is by induction on the dimension of X . We will first prove that there is a fibration $g : X \rightarrow \mathbb{P}^1$ since $n \geq 4$. We can reduce to the case when $n = 4$. The proof of four-dimensional case is given in section 6.

There is a Mori fibration $h : X \rightarrow Y$ such that $h \times g : X \rightarrow Y \times \mathbb{P}^1$ is a finite surjective morphism. We can prove further that Y also satisfies the condition in Theorem 1.5. By induction, we assume that Theorem 1.5 is true in dimension smaller than n . In particular, $h \times g$ induces a finite surjective morphism $f : X \rightarrow Z$

such that Z is either $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times W$, where W is a normal ample hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. In the end, we will conclude the proof of Theorem 1.5 by studying some finite morphisms between Fano varieties.

2. NOTATION OF THE PAPER

In this paper, we will work over \mathbb{C} , the field of complex numbers. A variety is an integral scheme of finite type defined over \mathbb{C} . If X is a variety, then we denote by $(X)^k$ the product of k copies of X . A fibration $f : X \rightarrow Y$ between normal varieties is a proper surjective morphism with connected fibers. The relative dimension of f is defined by $\dim f = \dim X - \dim Y$.

A normal variety X is said to be locally factorial (resp. \mathbb{Q} -factorial) if for any Weil divisor D in X , D (resp. some multiple of D) is a Cartier divisor. We refer to [KM98, §2.3] for the definition of klt singularities and canonical singularities. Note that locally factorial klt singularities are canonical. We denote by K_X a canonical divisor of X . The variety X is said to be Gorenstein if X is Cohen-Macaulay and K_X is a Cartier divisor. If D and E are two \mathbb{R} -divisors in X which are numerically equivalent, then we write $D \equiv E$. Let $\text{Pic}(X)$ be the Picard group of X and let $N^1(X)$ be the vector space $\text{Pic}(X) \otimes \mathbb{R} / \equiv$. We denote the real vector space of 1-cycles modulo numerical equivalence by $N_1(X)$. These two spaces $N_1(X), N^1(X)$ are dual to each other by the intersection pairing. The dimension $\rho(X)$ of $N^1(X)$ is called the Picard number of X . Let $\text{Nef}(X)$ be the cone of nef divisors in X . Let $\text{Psef}(X)$ be the cone of pseudo-effective divisors, that is, the closed convex cone generated by effective divisors. Then we have

$$\text{Nef}(X) \subseteq \text{Psef}(X) \subseteq N^1(X).$$

Let $\overline{\text{NE}}(X)$ be the Mori cone, that is, the closed convex cone in $N_1(X)$ generated by the classes of curves. Then $\overline{\text{NE}}(X)$ is dual to $\text{Nef}(X)$.

A normal variety X is said to be a Fano variety if its anti-canonical divisor $-K_X$ is a \mathbb{Q} -ample divisor, that is, some multiple of $-K_X$ is a very ample divisor. If X is a Fano variety with \mathbb{Q} -factorial klt singularities, then the cone $\overline{\text{NE}}(X)$ is polyhedral by the Cone Theorem (see [KM98, §3]).

3. DOUBLE COVERS BETWEEN NORMAL VARIETIES

We recall the notion of cyclic covers. Let Y be a normal variety and \mathcal{L} a line bundle on Y . Assume that there is a positive integer k and a section $s \in H^0(Y, \mathcal{L}^{\otimes k})$. Let D be the zero locus of s . Then there is a cyclic cover $g : Z \rightarrow Y$ with respect to the isomorphism $\mathcal{L}^{\otimes k} \cong \mathcal{O}_Y(D)$ induced by s . The morphism g is branched exactly along D and $g^*D = k \cdot \text{Supp}(g^*D)$ (see [KM98, Def. 2.50]). Note that, by construction, Z may not be normal. However, we have the following result.

Lemma 3.1. *Let Y be a normal Cohen-Macaulay variety and let D be a reduced divisor. Assume that there is a line bundle \mathcal{L} on Y such that $\mathcal{L}^{\otimes k} \cong \mathcal{O}_Y(D)$ for some $k > 0$. If $f : Z \rightarrow Y$ is the corresponding cyclic cover, then Z is normal and Cohen-Macaulay.*

Proof. We may assume that $Y = \text{Spec } A$ is affine. Then $Z = \text{Spec } A[T]/(T^k - s)$, where T is an indeterminate and $s = 0$ is the equation defining D . Since Y is Cohen-Macaulay, so is Z . Hence, we only need to prove that Z is smooth in codimension 1. However, there is an open subset Y_0 of Y such that $\text{codim } Y \setminus Y_0 \geq 2$ and both Y_0 and $D|_{Y_0}$ are smooth. Thus $Z_0 = g^{-1}(Y_0)$ is smooth and $\text{codim } Z \setminus Z_0 \geq 2$. This shows that Z is normal. \square

Lemma 3.2. *Let $f : X \rightarrow Y$ be an equidimensional fibration between normal Cohen-Macaulay varieties. Assume that there is an open subset U of Y whose complement has codimension at least 2 such that f has reduced fiber over U . Let $g : Y' \rightarrow Y$ be a cyclic cover with Y' normal. Let X' be the fiber product $X \times_Y Y'$. Then X' is a normal Cohen-Macaulay variety.*

Proof. Assume that g is the cyclic cover with respect to some isomorphism $\mathcal{L}^{\otimes k} \cong \mathcal{O}_Y(D)$, where D is a Cartier divisor and \mathcal{L} is a line bundle on Y . Since Y' is normal, D is reduced. Then the natural morphism $X' \rightarrow X$ is the cyclic cover with respect to the isomorphism $(f^*\mathcal{L})^{\otimes k} \cong \mathcal{O}_X(f^*D)$. Since f is equidimensional and has reduced fibers over U , f^*D is a reduced divisor in X . Since X is Cohen-Macaulay, by Lemma 3.1, X' is normal Cohen-Macaulay. \square

We will prove some properties on double covers.

Lemma 3.3. *Let $f : X \rightarrow Y$ be a double cover between normal varieties with Y locally factorial. Let D be the codimension 1 part of the discriminant of f . Then there is a line bundle \mathcal{L} on Y such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(D)$. If $g : Z \rightarrow Y$ is the corresponding cyclic cover, then X is the normalisation of Z . Moreover, if Y is Cohen-Macaulay, then X is smooth if and only if both D and Y are smooth.*

Proof. Let Y_0 be the largest open subset contained in the smooth locus of Y such that $X_0 = f^{-1}(Y_0)$ and $D|_{Y_0}$ are smooth. Then $\text{codim } Y \setminus Y_0 \geq 2$ and there is a line bundle \mathcal{L}_0 on Y_0 such that $f|_{X_0}$ is the cyclic cover with respect to $\mathcal{L}_0^{\otimes 2} \cong (\mathcal{O}_Y(D))|_{Y_0}$ (See [CD89]). Since Y is locally factorial, there is a line bundle \mathcal{L} on Y such that $\mathcal{L}|_{Y_0} \cong \mathcal{L}_0$. Then $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(D)$. Let $g : Z \rightarrow Y$ be the corresponding cyclic cover. Since X is normal, it is the normalisation of Z .

If Y is Cohen-Macaulay, then $X \cong Z$ by Lemma 3.1. Thus X is smooth if and only if D and Y are smooth (see [KM98, Lem. 2.51]). \square

Lemma 3.4. *Let $f : X \rightarrow Y$ be a double cover between normal varieties. Assume that Y is \mathbb{Q} -factorial. Let $D \subseteq Y$ be the codimension 1 part of the discriminant of f . Then X is Fano if and only if $-K_Y - \frac{1}{2}D$ is \mathbb{Q} -ample.*

Proof. Since Y is \mathbb{Q} -factorial, $-K_X$ is \mathbb{Q} -linearly equivalent to $-f^*(K_Y + \frac{1}{2}D)$. Since f is finite, we obtain that $-K_X$ is \mathbb{Q} -ample if and only if $-K_Y - \frac{1}{2}D$ is \mathbb{Q} -ample. \square

Lemma 3.5. *Let $f : X \rightarrow Y$ be an equidimensional Fano fibration of relative dimension 1 between normal projective varieties with \mathbb{Q} -factorial klt singularities. Then the relative Picard number of f is 1 if and only if f^*D is irreducible for every prime divisor D in Y .*

Proof. If the relative Picard number of f is 1, then f^*D is irreducible for every prime divisor D in Y .

Assume that the relative Picard number of f is at least 2. Since f is a Fano fibration and X has \mathbb{Q} -factorial klt singularities, we can run a MMP for X over Y by [BCHM10, Cor. 1.3.3]. We obtain the following sequence of birational maps over Y

$$\begin{array}{ccccccc} X_0 & \dashrightarrow & X_1 & \dashrightarrow & \cdots & \dashrightarrow & X_k \\ & \downarrow & \swarrow & & & \searrow & \\ & Y & & & & & \end{array}$$

such that $X = X_0$ and that X_k is a Mori fiber space. Let $X_k \rightarrow B$ be the Mori fibration. Then there is a natural morphism $B \rightarrow Y$. Since f has relative dimension 1, the natural morphism $B \rightarrow Y$ is birational. On the one hand, since f is equidimensional, for any prime divisor in X , its image by f has codimension at most 1 in Y . Hence, the morphism $B \rightarrow Y$ does not contract any divisor. On the other hand, since Y is \mathbb{Q} -factorial, the exceptional locus of $B \rightarrow Y$ is either empty or pure of codimension 1. Hence $B \rightarrow Y$ is an isomorphism.

Since the relative Picard number of X is larger than 1, there is some X_i in the previous sequence such that $X_i \rightarrow X_{i+1}$ is a divisorial contraction. In particular, if $f_i : X_i \rightarrow Y$ is the natural fibration, then there is a prime divisor D in Y such that f_i^*D is reducible. Hence f^*D is also reducible. \square

Lemma 3.6. *Let $g : Y \rightarrow Z$ be an equidimensional fibration between smooth projective varieties. Let $r : V \rightarrow Z$ be a Mori fibration with V smooth. Let X be the fiber product $Y \times_Z V$. Let $f : X \rightarrow Y$ and $\pi : X \rightarrow V$ be the natural fibrations.*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & V \\ f \downarrow & & \downarrow r \\ Y & \xrightarrow{g} & Z \end{array}$$

Assume that X is a smooth Fano variety and that the discriminants of g and r do not have common components. Then the relative Picard number of f is 1.

Proof. Assume the opposite. Then by Lemma 3.5, there is a prime divisor D in Y such that f^*D is reducible. Let $R = f^{-1}(D)$ and let R_1, R_2 be two different components of R . Let α be a general point in D . Let $E = g(D)$ and let $\beta = g(\alpha) \in E$. Then E is irreducible. Since g is equidimensional, the codimension of E is at most 1. Since $X = Y \times_Z V$, the fiber C' of r over β is isomorphic fiber C of f over α . Thus C' is

reducible for C is reducible. Since β is a general point in E , we obtain that E is a divisor and is contained in the discriminant of r .

Let $R' = r^{-1}(E)$. Then, by Lemma 3.5, it is irreducible since r is a Mori fibration. We have the following commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi|_R} & R' \\ f|_R \downarrow & & \downarrow r|_{R'} \\ D & \xrightarrow{g|_D} & E \end{array}$$

Since $X = Y \times_Z V$, the fibration π is equidimensional. Since R' is irreducible, we obtain that the projections $\pi|_{R_1} : R_1 \rightarrow R'$ and $\pi|_{R_2} : R_2 \rightarrow R'$ are surjective. In particular, general fibers of $\pi|_R$ are reducible.

Let a be a general point of R' and let $b = r(a) \in E$. Since $X = Y \times_Z V$, the fiber of $\pi|_R$ over a is the same as the fiber of $g|_D$ over b . Since the discriminants of g and r do not have common components and b is a general point on E , the fiber of $g|_D$ over b is irreducible. Hence the fiber of $\pi|_R$ over a is irreducible. This is a contradiction. \square

Thanks to this lemma, we can show that we can construct smooth Fano varieties X such that $\rho(X) = n - 1$ and $\text{Nef}(X) = \text{Psef}(X)$ with the methods in Example 1.3 and Example 1.4. For simplicity, we only look at the case of Example 1.4.2. The other cases are similar.

Construction 3.7. We will construct a smooth Fano variety X , of dimension n , with the methods of Example 1.4.2, such that $\rho(X) = n - 1$ and $\text{Nef}(X) = \text{Psef}(X)$.

Let W be a smooth divisor of degree $(1, k)$ in $B_1 \times B_2$, where $k \in \{1, 2\}$. Then the Picard number of W is 2 by the Lefschetz theorem (see [Laz04, Example 3.1.25]). We write

$$(\mathbb{P}^1)^{n-3} \times W = \mathbb{P}^1 \times T \text{ and } (\mathbb{P}^1)^{n-3} \times B_1 = \mathbb{P}^1 \times S.$$

There is a natural equidimensional fibration $T \rightarrow S$. We can choose a smooth double cover $Y_1 \rightarrow \mathbb{P}^1 \times S$, branched along some smooth divisor D of degree $(2, \dots, 2, 2)$, such that the discriminant of $Y_1 \rightarrow S$ and the discriminant of $T \rightarrow S$ do not have common components. Moreover, we choose D such that q^*D is smooth, where $q : \mathbb{P}^1 \times T \rightarrow \mathbb{P}^1 \times S$ is the natural fibration.

Let X be the pullback of $(\mathbb{P}^1)^{n-3} \times W$ by $Y_1 \times B_2 \rightarrow (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Then we have

$$X \cong ((\mathbb{P}^1)^{n-3} \times W) \times_{(\mathbb{P}^1)^{n-3} \times B_1} Y_1 = (\mathbb{P}^1 \times T) \times_{\mathbb{P}^1 \times S} Y_1 \cong T \times_S Y_1.$$

Now we will show that $\rho(X) = n - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. By [Dru14, Lem. 3.4], the Picard number of Y_1 is the same as the Picard number of $\mathbb{P}^1 \times S$. Thus the natural projection $Y_1 \rightarrow S$ is a Mori fibration. Note that the natural morphism $p : X \rightarrow \mathbb{P}^1 \times T$ is a double cover branched along q^*D . Since q^*D and $\mathbb{P}^1 \times T$ are smooth, X is smooth by Lemma 3.3. Moreover, X is a Fano variety by Lemma 3.4. Hence, by Lemma 3.6, $X \rightarrow T$ has relative Picard 1. Since T has Picard number $n - 2$, we have $\rho(X) = n - 1$.

Let E be a pseudo-effective divisor in X . Then p_*E is a pseudo-effective divisor in $\mathbb{P}^1 \times T$. Note that the nef cone of $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$ is equal to the pseudo-effective cone of $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. By the Lefschetz theorem (see [Laz04, Example 3.1.25]), the nef cone of $\mathbb{P}^1 \times T$ is equal to the pseudo-effective cone of $\mathbb{P}^1 \times T$. Thus p_*E is nef. Hence E is also a nef divisor in X by the projection formula. This shows that $\text{Nef}(X) = \text{Psef}(X)$.

4. FIBRATIONS ON VARIETIES OF THEOREM 1.5

Let X be a \mathbb{Q} -factorial klt Fano variety. Then for every face V of $\overline{\text{NE}}(X)$, there is a fibration $f : X \rightarrow Y$ which contracts exactly the curves whose classes are in V . Conversely, if $f : X \rightarrow Y$ is a fibration, then the curves in the fibers of f generates a face V of $\overline{\text{NE}}(X)$ (See for example [KMM87, §3-2]).

Lemma 4.1. *Let X be a Fano variety with \mathbb{Q} -factorial klt (resp. locally factorial canonical) singularities such that $\text{Nef}(X) = \text{Psef}(X)$. Let $f : X \rightarrow Y$ be any fibration. Then Y is also a Fano variety with \mathbb{Q} -factorial klt (resp. locally factorial canonical) singularities such that $\text{Nef}(Y) = \text{Psef}(Y)$.*

Proof. By [Dru14, Lem. 4.2], we have $\text{Nef}(Y) = \text{Psef}(Y)$. If X is with locally factorial canonical singularities, then so is Y by [Dru14, Cor. 4.8]. If X has \mathbb{Q} -factorial klt singularities, then the same argument of the proof

of [Dru14, Cor. 4.8] shows that Y also has \mathbb{Q} -factorial klt singularities. By [FG12, Thm. 3.1], $-K_Y$ is big. Since $\text{Nef}(Y) = \text{Psef}(Y)$, this implies that $-K_Y$ is ample. \square

The objective of this section is to prove the two following lemmas.

Lemma 4.2. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. Assume that there is no fibration from X to \mathbb{P}^1 . Let $f : X \rightarrow Y$ be a fibration. Then Y is also a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\text{Nef}(Y) = \text{Psef}(Y)$.*

Lemma 4.3. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. Assume that there is a fibration $f_1 : X \rightarrow W$ and a projective morphism $f_2 : X \rightarrow (\mathbb{P}^1)^r$ such that the product $f_1 \times f_2 : X \rightarrow W \times (\mathbb{P}^1)^r$ is a finite surjective morphism. Then W is also a Fano variety with locally factorial canonical singularities such that W is smooth in codimension 2, $\rho(W) = \dim W - 1$ and $\text{Nef}(W) = \text{Psef}(W)$.*

We will first prove some preliminary results.

Lemma 4.4. *Let X be a \mathbb{Q} -factorial klt Fano variety such that $\text{Nef}(X) = \text{Psef}(X)$. Let $f : X \rightarrow Y$ be a fibration. If the relative dimension of f is 1, then f is a Mori fibration.*

Proof. There is a Mori fibration $X \rightarrow Z$ over Y by [Dru14, Lem. 4.4]. The variety Z is a \mathbb{Q} -factorial klt Fano variety such that $\text{Nef}(Z) = \text{Psef}(Z)$ by Lemma 4.1. Since f has relative dimension 1, the natural fibration $Z \rightarrow Y$ is birational. Since $\text{Nef}(Z) = \text{Psef}(Z)$, we have $Z \cong Y$ by [Dru14, Lem. 4.4]. This implies that f is a Mori fibration. \square

Lemma 4.5. *Let $f : X \rightarrow Y$ be a Mori fibration such that X has locally factorial canonical singularities. If X is smooth in codimension k and f is equidimensional of relative dimension 1, then Y is also smooth in codimension k .*

Proof. Let $y \in Y$ be a point. The problem is local around y . Without loss of generality, we may assume that Y is affine. Since f is a Fano fibration of relative dimension 1, by the main theorem of [AW93], the linear system $|-K_X|$ is basepoint-free. Let Z be a general divisor in the linear system $|-K_X|$. Then Z is smooth in codimension k . Moreover, since f is equidimensional, by shrinking Y if necessary, we may assume further that the morphism $f|_Z : Z \rightarrow Y$ is finite surjective of degree 2.

Note that Y has klt singularities (see [Fuj99, Cor. 3.5]). Thus it is Cohen-Macaulay. Moreover, it is locally factorial (see for example [Dru14, Lem. 4.6]). Since Z is smooth in codimension k , by Lemma 3.3, Y is smooth in codimension k . \square

Lemma 4.6. *Let X be a Fano variety with \mathbb{Q} -factorial klt singularities such that $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. Let $f : X \rightarrow Y$ be a Mori fibration. Then the dimension of Y is equal to $\rho(Y)$ or $\rho(Y) + 1$.*

Proof. By Lemma 4.1, Y is a \mathbb{Q} -factorial klt Fano variety with $\text{Nef}(Y) = \text{Psef}(Y)$. In particular, we have $\rho(Y) \leq \dim Y$ by [Dru14, Lem. 4.9]. Since f is a Mori fibration, we have $\rho(Y) = \rho(X) - 1$. Hence $\dim Y \leq \dim X - 1 = \rho(X) = \rho(Y) + 1$. Thus $\dim Y$ is equal to $\rho(Y)$ or $\rho(Y) + 1$. \square

Lemma 4.7. *Let X be a \mathbb{Q} -factorial klt Fano variety such that $\text{Nef}(X) = \text{Psef}(X)$ and $\rho(X) = \dim X$. Then there is a fibration from X to \mathbb{P}^1 .*

Proof. Let V be a face in $\overline{\text{NE}}(X)$ of codimension 1 and let $f : X \rightarrow B$ be the corresponding contraction. Then $\dim B = \rho(B) = 1$ by [Dru14, Lem. 4.9.2]. This shows that $B \cong \mathbb{P}^1$. \square

Lemma 4.8. *Let X be a Fano variety with \mathbb{Q} -factorial klt singularities such that $\rho(X) = \dim X - 1 \geq 2$ and $\text{Nef}(X) = \text{Psef}(X)$. Assume that there is no fibration from X to \mathbb{P}^1 . Then every Mori fibration $f : X \rightarrow Y$ is equidimensional of relative dimension 1.*

Proof. We will prove the lemma by induction on the dimension of X . If $\dim X = 3$, then $\dim Y = 2$. In this case, the fibration f is equidimensional. Assume that the lemma is true if $\dim X = n$ for some $n \geq 3$.

Now we assume that $\dim X = n + 1$. Suppose that there is a fiber F of f such that $\dim F \geq 2$. Assume that f is the fibration which corresponds to an extremal ray R of $\overline{\text{NE}}(X)$. Let R_1 be another extremal ray such

that R and R_1 generates a face V of dimension 2. Let $g : X \rightarrow Z$ be the Mori fibration which corresponds to R_1 and let $X \rightarrow W$ be the fibration corresponding to V . We obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow h \\ Z & \xrightarrow{p} & W \end{array}$$

By Lemma 4.1, Z is a \mathbb{Q} -factorial klt Fano variety such that $\text{Nef}(Z) = \text{Psef}(Z)$. By Lemma 4.6, $\rho(Z)$ is either $\dim Z$ or $\dim Z - 1$. Since there is no fibration from Z to \mathbb{P}^1 , we have $\rho(Z) = \dim Z - 1$ by Lemma 4.7. Since $\rho(Z) = \rho(X) - 1 = n - 1$, we obtain that $\dim Z = n$. By induction hypothesis, the morphism p is equidimensional of relative dimension 1. Since none of the curves in F is contracted by g , the dimension of $E = g(F)$ is also larger than 1. Moreover, since $h(f(F))$ is a point, $p(E)$ is also a point. We obtain a contradiction. This completes the induction and the proof of the lemma. \square

Now we will prove Lemma 4.2 and Lemma 4.3.

Proof of Lemma 4.2. We will prove the lemma by induction on the relative Picard number k of f . If f is a Mori fibration, then by Lemma 4.8, f is equidimensional of relative dimension 1. Hence $\rho(Y) = \dim(Y) - 1$. Moreover, Y is a Fano variety with locally factorial canonical singularities such that $\text{Nef}(Y) = \text{Psef}(Y)$ by Lemma 4.1 and is smooth in codimension 2 by Lemma 4.5.

Assume that the lemma is true for $k \leq l$ for some $l \geq 1$. Now we consider the case when $k = l + 1$. There is a Mori fibration $g : X \rightarrow Z$ over Y since $\text{Nef}(X) = \text{Psef}(X)$ (see [Dru14, Lem. 4.4]). Hence, by induction, Z is a Fano variety with locally factorial canonical singularities such that Z is smooth in codimension 2, $\rho(Z) = \dim Z - 1$ and $\text{Nef}(Z) = \text{Psef}(Z)$. Moreover, the relative Picard number of $Z \rightarrow Y$ is l . Thus, by induction hypotheses, Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\text{Nef}(Y) = \text{Psef}(Y)$. This completes the proof of the Lemma. \square

Proof of Lemma 4.3. We will prove the lemma by induction on r . If $r = 1$, then the morphism f_1 is equidimensional of relative dimension 1 since $f_1 \times f_2$ is finite. By Lemma 4.4, f_1 is a Mori fibration. By Lemma 4.1 and Lemma 4.5, we obtain that W is a Fano variety with locally factorial canonical singularities such that W is smooth in codimension 2, $\rho(W) = \dim W - 1$ and $\text{Nef}(W) = \text{Psef}(W)$. Assume that the lemma is true for $r \leq k$ where $k \geq 1$ is an integer.

Now we assume that $r = k + 1$. We write $(\mathbb{P}^1)^{k+1} = (\mathbb{P}^1)^k \times \mathbb{P}^1$ and let $X \rightarrow Y$ be the Stein factorisation of the morphism $X \rightarrow W \times (\mathbb{P}^1)^k$. Then the product $X \rightarrow Y \times \mathbb{P}^1$ is also a finite surjective morphism. Hence, by induction hypotheses, Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\text{Nef}(Y) = \text{Psef}(Y)$. There is a natural fibration $g_1 : Y \rightarrow W$ induced by $X \rightarrow W$. There is a morphism $g_2 : Y \rightarrow (\mathbb{P}^1)^k$ induced by $X \rightarrow (\mathbb{P}^1)^k$. Since $X \rightarrow W \times (\mathbb{P}^1)^k \times \mathbb{P}^1$ is finite surjective, we obtain that $g_1 \times g_2 : Y \rightarrow W \times (\mathbb{P}^1)^k$ is also finite surjective. By induction hypotheses, W is a Fano variety with locally factorial canonical singularities such that W is smooth in codimension 2, $\rho(W) = \dim W - 1$ and $\text{Nef}(W) = \text{Psef}(W)$. This completes the proof of the lemma. \square

5. FINITE MORPHISMS BETWEEN FANO THREEFOLDS

In this section, we will prove some results on finite morphisms between Fano threefolds. As a corollary, we prove Theorem 1.2. Recall that if X is a Fano threefold with Gorenstein canonical singularities, then

$$h^0(X, \mathcal{O}_X(-K_X)) = -\frac{1}{2}K_X^3 + 3.$$

In particular, $-K_X^3$ is a positive even integer (see [Rei83, §4.4]). We will first prove some lemmas.

Lemma 5.1. *Let $f : X \rightarrow Y$ be a Fano fibration of relative dimension 1 between normal quasi-projective varieties. Assume that X is smooth in codimension 2. Then there is an open subset U of Y whose complement has codimension at least 2 such that f has reduced fibers over U .*

Proof. By taking general hyperplane sections in Y , we can reduce to the case when X is a smooth surface and Y is a smooth curve.

Let y be a point in Y and let $E = f^*y$. Then $-K_X \cdot E = 2$. Since K_X is Cartier and f is a Fano fibration, this implies that E has at most two components. If E has two components C and D , then both of them are reduced. If E has one component C , then E is reduced for f has a section by the Tsen's theorem. \square

Lemma 5.2. *Let X be a normal threefold and let $B_1 \cong B_2 \cong \mathbb{P}^2$. Assume that there are two equidimensional fibrations $f_1 : X \rightarrow B_1$ and $f_2 : X \rightarrow B_2$ such that $f_1 \times f_2 : X \rightarrow B_1 \times B_2$ is finite onto its image W . Assume that W is of degree (p, q) such that $p, q \in \{1, 2\}$. Then W is normal.*

Proof. Since W is Cohen-Macaulay, we only need to prove that it is smooth in codimension 1. Let Z be the singular locus of W . Let $p_i : W \rightarrow B_i$ be the natural projection for $i = 1, 2$. Then p_1 is equidimensional. Since W is Cohen-Macaulay and B_1 is smooth, this implies that p_1 is flat.

Since general fibers of $f_1 : X \rightarrow B_1$ are irreducible, general fibers of p_1 are irreducible. Since W is reduced, general fibers of p_1 are reduced. Since $q \in \{1, 2\}$, we obtain that general fibers of p_1 are smooth rational curves. Since p_1 is flat, this implies that $p_1(Z)$ is a proper subvariety of B_1 .

By symmetry, we can obtain that $p_2(Z)$ is also a proper subvariety of B_2 . Thus $\text{codim } Z \geq 2$. This completes the proof of the lemma. \square

Lemma 5.3. *Let W be an ample normal hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. Assume that W is a Fano variety with \mathbb{Q} -factorial klt singularities. Then the natural morphisms from W to \mathbb{P}^2 are equidimensional.*

Proof. Since W is ample, it has Picard number 2 by the Lefschetz theorem (see [Laz04, Example 3.1.25]). Thus, if $g : W \rightarrow \mathbb{P}^2$ is a natural projection, then it is a Mori fibration. Since W is a \mathbb{Q} -factorial klt threefold, this implies that g is equidimensional. \square

Lemma 5.4. *Let $f : X \rightarrow Y$ be a finite surjective morphism between normal \mathbb{Q} -factorial varieties. Assume that $-K_X$ is big. Then $-K_Y$ is a big divisor.*

Proof. See [Dru14, Lem. 8.1]. \square

Lemma 5.5. *Let $f : X \rightarrow Y$ be an equidimensional Fano fibration of relative dimension 1 such that X has Gorenstein canonical singularities and Y is smooth. Then f is a conic bundle. That is, at least locally on Y , X can be embedded in $\mathbb{P}^2 \times Y$ such that every fiber of f is a conic in \mathbb{P}^2 .*

Proof. Since $-K_X$ is relatively ample, $R^i f_*(\mathcal{O}_X(-K_X)) = 0$ for all $i > 0$ by the Kawamata-Viehweg vanishing theorem (see [KMM87, Thm. 1-2-5]). Since f is equidimensional of relative dimension 1, for every $y \in Y$, we have $h^2(X_y, \mathcal{O}_{X_y}(-K_X|_{X_y})) = 0$. By [Har77, Thm. III.12.11], we obtain that, for all $y \in Y$,

$$R^1 f_*(\mathcal{O}_X(-K_X)) \otimes k(y) \rightarrow H^1(X_y, \mathcal{O}_{X_y}(-K_X|_{X_y}))$$

is an isomorphism, where $k(y)$ is the residue field of y . Hence

$$h^1(X_y, \mathcal{O}_{X_y}(-K_X|_{X_y})) = 0$$

for all $y \in Y$. By [Har77, Thm. III.12.11] again, we obtain that

$$f_*(\mathcal{O}_X(-K_X)) \otimes k(y) \rightarrow H^0(X_y, \mathcal{O}_{X_y}(-K_X|_{X_y}))$$

is an isomorphism for all $y \in Y$.

Note that general fibers of f are smooth rational curves. Hence the Euler characteristic of the restriction of $\mathcal{O}_X(-K_X)$ on a general fiber of f is 3. Since f is equidimensional, X is Cohen-Macaulay and Y is smooth, the morphism f is flat. Since $\mathcal{O}_X(-K_X)$ is locally free on X , it is flat over Y . Thus the Euler characteristic of the restriction of $\mathcal{O}_X(-K_X)$ on every fiber of f is 3 (see the proof of [Har77, Thm. III.9.9]). This shows that $f_*(\mathcal{O}_X(-K_X))$ is a locally free sheaf of rank 3 by [Har77, Cor. III.12.9].

The problem is local on Y . Thus, we may assume that Y is affine and $f_*(\mathcal{O}_X(-K_X))$ is a free sheaf of rank 3. By the main theorem of [AW93], $\mathcal{O}_X(-K_X)$ is f -relatively generated. Since the sheaf $f_*(\mathcal{O}_X(-K_X))$ has rank 3, $f^* f_*(\mathcal{O}_X(-K_X))$ induces a morphism, over Y ,

$$\varphi : X \rightarrow \mathbb{P}^2 \times Y.$$

Since $f_*(\mathcal{O}_X(-K_X)) \otimes k(y) \rightarrow H^0(X_y, \mathcal{O}_{X_y}(-K_X|_{X_y}))$ is an isomorphism and $-K_X|_{X_y}$ is very ample for general $y \in Y$, the morphism φ is birational onto its image. Since $-K_X$ is f -relatively ample, the morphism φ is in fact an isomorphism onto its image.

Since $-K_X$ has degree 2 on every fiber of f , we obtain that $\varphi(X_y) \subseteq \mathbb{P}^2 \times \{y\}$ is a conic for every $y \in Y$. This completes the proof of the lemma. \square

Lemma 5.6. *Let Z_1, Z_2 be two surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let Z be a normal hypersurface of degree $(1, 1, 1, 1)$ in $Z_1 \times Z_2$. Assume that there is a finite morphism $f : X \rightarrow Z$ of degree $d \leq 2$ such that*

- X is Fano threefold with Gorenstein canonical singularities;
- the natural morphisms $f_1 : X \rightarrow Z_1$ and $f_2 : X \rightarrow Z_2$ induced by f are equidimensional fibrations;
- $K_X \equiv \frac{1}{d}f^*K_Z$.

Then f is an isomorphism.

Proof. Assume that f is of degree 2. There are two natural morphisms from Z to $\mathbb{P}^1 \times Z_2$ induced by the natural projections from $Z_1 \times Z_2$ to $\mathbb{P}^1 \times Z_2$. Let $\varphi : Z \rightarrow \mathbb{P}^1 \times Z_2$ be one of them. Since Z is of degree $(1, 1, 1, 1)$, we obtain that φ is birational. Moreover, φ is not an isomorphism since the Picard number of Z is 4 by the Lefschetz theorem ([Laz04, Example 3.1.25]). Thus its exceptional locus is non empty. Moreover, it is pure of codimension 1 since $\mathbb{P}^1 \times Z_2$ is smooth. Hence the morphism φ contracts at least one divisor. Since the natural fibration $g_2 : Z \rightarrow B_2$ factors through φ , we obtain that there is a prime divisor H in Z_2 such that g_2^*H is reducible.

The fibration g_2 is equidimensional since f_2 is. Every fiber of g_2 is isomorphic to a divisor of degree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Hence it is either a smooth rational curve or the union of two smooth rational curves which intersect at one point. This shows that g_2^*H has two components R_1, R_2 and every fiber of g_2 over H has two components.

Let D be the codimension 1 part of the discriminant of $f : X \rightarrow Z$. Then K_X is linearly equivalent to $f^*(K_Z) + f^{-1}(D)$. Since K_X and K_Z are Cartier divisors, we obtain that $f^{-1}(D)$ is Cartier. Since f is finite, this implies that D is \mathbb{Q} -Cartier. Then K_X is \mathbb{Q} -linearly equivalent to $f^*(K_Z + \frac{1}{2}D)$. Thus we have $D \equiv -K_Z$. In particular, D is \mathbb{Q} -ample and the intersection number of any fiber of g_2 and D is equal to 2.

The divisor f_2^*H has at least two components. Since X has Gorenstein canonical singularities and f_2 is equidimensional of relative dimension 1, every fiber of f_2 has at most two components by Lemma 5.5. We obtain that f_2^*H also has two components S_1, S_2 . Assume that $f(S_1) = R_1$ and $f(S_2) = R_2$. Let b be any point on H and let G be the fiber of g_2 over b . Then $G = G_1 \cup G_2$, where $G_1 = G \cap R_1$ and $G_2 = G \cap R_2$. By Lemma 5.5 again, the fiber F of f_2 over b is reduced and is the union of two smooth rational curves F_1 and F_2 which meet at one point. In particular, G is not contained in D . We assume that $F_i \subseteq S_i$ for $i = 1, 2$.

Both $f|_{F_1} : F_1 \rightarrow G_1$ and $f|_{F_2} : F_2 \rightarrow G_2$ are finite morphism of degree 2. Since $G_1 \setminus G_2$ is simply connected, it contains a point which is in the branch locus of $f|_{F_1} : F_1 \rightarrow G_1$. Since g_2 is flat, $G_1 \setminus G_2$ is contained in Z_{ns} , the smooth locus of Z . Moreover, over Z_{ns} , the morphism f is branched exactly along $Z_{ns} \cap D$ by the Zariski purity theorem (see [Zar58, Prop. 2]). Thus D meets $G_1 \setminus G_2$ at at least one point. By symmetry, D meets $G_2 \setminus G_1$ at at least one point. Since $D \cdot G = 2$, we obtain that G and D meet at exactly two points which are smooth points of G .

Since b is chosen arbitrarily, we obtain that $D \cap R_1 \cap R_2$ is empty. However, since $R_1 \cap R_2$ is a subscheme of dimension 1 and D is \mathbb{Q} -ample, $D \cap R_1 \cap R_2$ is not empty. We obtain a contradiction. \square

5.1. Finite covers of $\mathbb{P}^1 \times \mathbb{P}^2$.

Proposition 5.7. *Let X be a Fano threefold with Gorenstein canonical singularities. Assume that there are two fibrations $f_1 : X \rightarrow \mathbb{P}^1$ and $f_2 : X \rightarrow \mathbb{P}^2$ such that $f = f_1 \times f_2 : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is finite. Assume that K_X is numerically equivalent to the pullback of some \mathbb{Q} -divisor in $\mathbb{P}^1 \times \mathbb{P}^2$ by f . Then $d = \deg f$ is at most 2. Moreover, if $d = 2$, then f is a double cover which is branched along a divisor of degree $(2, k)$, where $k \in \{2, 4\}$.*

Proof. Let A_1, A_2 be two divisors in X such that $\mathcal{O}_X(A_1) \cong f_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_X(A_2) \cong f_2^*\mathcal{O}_{\mathbb{P}^2}(1)$. By assumption, there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1A_1 + a_2A_2$.

We will first prove that $d \leq 2$. Note that

$$A_1A_2^2 = d \text{ and } A_1^2A_2 = A_1^3 = A_2^3 = 0.$$

Since f_2 is a Fano fibration, general fibers of f_2 are smooth rational curves. This implies that $-K_X \cdot A_2^2 = 2$. Hence, $a_1 = \frac{2}{d}$. Moreover, we have

$$(-K_X)^3 = 3a_1a_2^2A_1A_2^2 = 3a_1a_2^2d = 6a_2^2.$$

Since $-K_X$ is a Cartier divisor, we obtain that $6a_2^2 \in \mathbb{Z}$. Hence $a_2 \in \mathbb{Z}$.

We have $(-K_X - a_2A_2) \equiv \frac{2}{d}A_1$. Hence the Cartier divisor $-K_X - a_2A_2$ is numerically trivial on every fiber of f_1 . Since f_1 is a Fano fibration, we obtain that $-K_X - a_2A_2$ is linearly equivalent to some integral multiple of A_1 by [KMM87, Lem. 3-2-5]. Thus $\frac{2}{d} \in \mathbb{Z}$ and $d \leq 2$.

If $d = 1$, then $X \cong \mathbb{P}^1 \times \mathbb{P}^2$. If $d = 2$, then assume that f is branched along a divisor D of degree (p, q) . Then by Lemma 3.4, X is Fano if and only if $p \leq 3$ and $q \leq 5$. Since f is a double cover, by Lemma 3.3, there is a line bundle \mathcal{L} on $\mathbb{P}^1 \times \mathbb{P}^2$ such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}(D)$. Hence p, q are even. Moreover, since f_1, f_2 have connected fibers, we have $p, q > 0$. Hence f is branched along a divisor of degree $(2, 2)$ or $(2, 4)$. This completes the proof of the proposition. \square

Lemma 5.8. *Let X be a Fano threefold with Gorenstein canonical singularities. Assume that there is a fibration $f_1 : X \rightarrow \mathbb{P}^1$ and a projective surjective morphism $f_2 : X \rightarrow \mathbb{P}^2$ such that general fibers of f_2 have two connected components. Assume that $f = f_1 \times f_2 : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is finite. Let A_1, A_2 be two divisors in X such that $\mathcal{O}_X(A_1) \cong f_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_X(A_2) \cong f_2^* \mathcal{O}_{\mathbb{P}^2}(1)$. Assume that there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1 A_1 + a_2 A_2$. Then f is of degree 2 or 4 and $a_1, a_2 \in \mathbb{Z}$.*

Proof. Since general fibers of f_2 have two connected components, the degree of f is an even integer. Set $\deg f = 2d$. Then we have

$$A_1 A_2^2 = 2d \text{ and } A_1^2 A_2 = A_1^3 = A_2^3 = 0.$$

Note that each component of a general fiber of f_2 is a smooth rational curve. This implies that

$$-K_X \cdot A_2^2 = 2 \times 2 = 4.$$

Hence, $a_1 = \frac{2}{d}$. Moreover, we have

$$(-K_X)^3 = 3a_1 a_2^2 A_1 A_2^2 = 6a_1 a_2^2 d = 12a_2^2.$$

Since $(-K_X)^3$ is an even integer, we obtain that $6a_2^2 \in \mathbb{Z}$. This implies that $a_2 \in \mathbb{Z}$.

We have $(-K_X - a_2 A_2) \equiv \frac{2}{d} A_1$. As in the proof of Proposition 5.7, we obtain that $\frac{2}{d} \in \mathbb{Z}$. Hence $d \leq 2$ and $a_1 \in \mathbb{Z}$. \square

5.2. Finite covers of hypersurfaces of $\mathbb{P}^2 \times \mathbb{P}^2$.

Proposition 5.9. *Let X be a Fano threefold with Gorenstein canonical singularities. Assume that there are two equidimensional fibrations $f_1 : X \rightarrow B_1$ and $f_2 : X \rightarrow B_2$ such that $B_1 \cong B_2 \cong \mathbb{P}^2$ and that $f_1 \times f_2 : X \rightarrow B_1 \times B_2$ is finite onto its image W . Assume that K_X is numerically equivalent to the pullback of some \mathbb{Q} -divisor in $B_1 \times B_2$ by $f_1 \times f_2$. Let $f : X \rightarrow W$ be the natural morphism. Then there are exactly four possibilities:*

- (1) f is an isomorphism, X is a hypersurface of degree $(1, 1)$;
- (2) f is an isomorphism, X is a hypersurface of degree $(2, 1)$ or $(1, 2)$;
- (3) f is an isomorphism, X is a hypersurface of degree $(2, 2)$;
- (4) W is a smooth hypersurface of degree $(1, 1)$, f is a double cover branched along a divisor which is the intersection of W and a divisor of degree $(2, 2)$ in $B_1 \times B_2$.

Recall that a divisor of degree $(2, 2)$ in W

Proof. Let A_1, A_2 be two divisors in X such that $\mathcal{O}_X(A_1) \cong f_1^* \mathcal{O}_{B_1}(1)$ and $\mathcal{O}_X(A_2) \cong f_2^* \mathcal{O}_{B_2}(1)$. By assumption, there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1 A_1 + a_2 A_2$. Set $d = \deg f$. Assume that W is of degree (p, q) in $B_1 \times B_2$. We have

$$A_1^2 A_2 = dq \text{ and } A_1 A_2^2 = dp.$$

Moreover, $A_1^3 = A_2^3 = 0$. Since general fibers of f_1 and f_2 are smooth rational curves, we have $-K_X \cdot A_1^2 = -K_X \cdot A_2^2 = 2$. Hence

$$a_1 = \frac{2}{dp} \text{ and } a_2 = \frac{2}{dq}.$$

We have

$$-K_X^3 = 3a_1 a_2^2 A_1 A_2^2 + 3a_1^2 a_2 A_1^2 A_2 = \frac{24}{d^2} \left(\frac{1}{q^2} + \frac{1}{p^2} \right).$$

Since $-K_X^3$ a positive even integer, the number

$$r = \frac{12}{d^2} \left(\frac{1}{q^2} + \frac{1}{p^2} \right)$$

is a positive integer. Without loss of generality, we may assume that $q \leq p$.

If $d = 1$, then r is an integer if and only if $q, p \in \{1, 2\}$. The morphism $f : X \rightarrow W$ is the normalisation map. However, W is normal by Lemma 5.2. Thus f is an isomorphism.

If $d = 2$, then we can only have $q = p = 1$. Since $f_1 : X \rightarrow B_1$ is equidimensional, so is the induced projection $p_1 : W \rightarrow B_1$. Hence every fiber of p_1 is a line in \mathbb{P}^2 , which is smooth. Since W is Cohen-Macaulay and \mathbb{P}^2 is smooth, the morphism p_1 is flat. Hence p_1 is a smooth morphism and W is smooth. Since W is ample in $\mathbb{P}^2 \times \mathbb{P}^2$, the natural morphism $\text{Pic}(\mathbb{P}^2 \times \mathbb{P}^2) \rightarrow \text{Pic}(W)$ is an isomorphism by the Lefschetz theorem (see [Laz04, Example 3.1.25]). Similarly to the last paragraph of the proof of Proposition 5.7, we obtain that f is branched along a divisor which is the intersection of W and a divisor of degree $(2, 2)$ in $B_1 \times B_2$.

If $d \geq 3$, then we have

$$1 \leq r \leq \frac{4}{3} \left(\frac{1}{q^2} + \frac{1}{p^2} \right).$$

Hence either $q = 1$ or $q, p \leq 2$. If $q = p = 1$, then $r = \frac{24}{d^2}$. In this case, r is an integer if and only if $d = 1$ or 2 . This is impossible since $d > 2$. If $q = 1$ and $p \geq 2$, then the previous inequality implies that $r \leq \frac{5}{3}$. Thus $r = 1$ and p^2 divides 12. In this case, we can only have $p = 2$ and $1 = r = \frac{12}{d^2} \cdot \frac{5}{4} = \frac{15}{d^2}$. This is impossible. If $q = p = 2$, then $r = \frac{6}{d^2}$ is an integer if and only if $d = 1$. This is also a contradiction.

Hence $d \leq 2$. This completes the proof of the proposition. \square

Lemma 5.10. *Let X be a Fano threefold with Gorenstein canonical singularities. Let $B_1 \cong B_2 \cong \mathbb{P}^2$. Assume that there are two equidimensional projective morphisms $f_1 : X \rightarrow B_1$ and $f_2 : X \rightarrow B_2$ such that general fibers of f_i have two connected components for $i = 1, 2$ and $f_1 \times f_2 : X \rightarrow B_1 \times B_2$ is finite onto its image W . Let $g_i : X \rightarrow Z_i \rightarrow B_i$ be the Stein factorisation of f_i for $i = 1, 2$. Let A_1, A_2 be two divisors in X such that $\mathcal{O}_X(A_1) \cong f_1^* \mathcal{O}_{B_1}(1)$ and $\mathcal{O}_X(A_2) \cong f_2^* \mathcal{O}_{B_2}(1)$. Assume that*

- W is a normal hypersurface with isolated \mathbb{Q} -factorial klt singularities of degree (p, q) , where $p, q \in \{1, 2\}$;
- Z_1 and Z_2 are smooth;
- there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1 A_1 + a_2 A_2$.

Let $f : X \rightarrow W$ be the natural morphism. Then the degree of f is 4 and $a_1 = a_2 = p = q = 1$.

Proof. Let h be the natural morphism $Z_1 \times Z_2 \rightarrow B_1 \times B_2$ and let $Z = h^*W$. We will first show that that Z is normal. Since Z_1 is smooth, the double cover $Z_1 \rightarrow B_1$ is a cyclic cover with respect to some isomorphism $(\mathcal{L}_1)^{\otimes 2} \cong \mathcal{O}_{B_1}(D_1)$, where D_1 is a smooth divisor (see Lemma 3.3). Let $p_1 : W \rightarrow B_1$ be the natural projection. Let $V \subseteq Z_1 \times B_2$ be the pullback of W by $Z_1 \times B_2 \rightarrow B_1 \times B_2$. Then $V \cong W \times_{B_1} Z_1$.

Since W is a Fano threefold with \mathbb{Q} -factorial klt singularities, p_1 is an equidimensional Fano fibration by Lemma 5.3. Moreover, since W has isolated singularities, by Lemma 5.1, we obtain that p_1 has reduced fibers over some open subset of B_1 whose complement has codimension at least 2. Thus V is normal and Cohen-Macaulay by Lemma 3.2.

Let $p_2 : W \rightarrow B_2$ and $r_2 : V \rightarrow B_2$ be the natural projections. Then there is an open subset U of B_2 such that $\text{codim } B_2 \setminus U \geq 2$ and p_2 has reduced fibers over U by Lemma 5.3. Note that $V \rightarrow W$ is branched along $p_1^* D_1 \subseteq W$ and every component of $p_1^* D_1 \subseteq W$ is horizontal over B_2 . Hence there is an open subset U' of U such that $\text{codim } B_2 \setminus U' \geq 2$ and r_2 has reduced fibers over U' . Since Z is the pullback of V by $Z_1 \times Z_2 \rightarrow Z_1 \times B_2$, we have $Z \cong V \times_{B_2} Z_2$. As in the previous paragraph, Lemma 3.2 implies that Z is normal.

Hence Z is the image of X in $Z_1 \times Z_2$. Let $g : X \rightarrow Z$ be the natural morphism. Let $d = \deg f$. Since $Z \rightarrow W$ is of degree 4, we obtain that $d \in 4\mathbb{Z}$. We have

$$A_1^2 A_2 = qd \text{ and } A_1 A_2^2 = pd.$$

Note that every component of general fibers of f_i is a smooth rational curve for $i = 1, 2$. Hence we have

$$-K_X A_1^2 = -K_X A_2^2 = 4.$$

This shows that

$$a_1 = \frac{4}{pd}, \text{ and } a_2 = \frac{4}{qd}.$$

We have

$$-K_X^3 = 3a_1 a_2^2 A_1 A_2^2 + 3a_1^2 a_2 A_1^2 A_2 = \frac{192}{d^2} \left(\frac{1}{p^2} + \frac{1}{q^2} \right).$$

Since $-K_X^3$ is a positive even integer, we obtain that

$$r = \frac{96}{d^2} \left(\frac{1}{p^2} + \frac{1}{q^2} \right)$$

is an integer. Since $d \in 4\mathbb{Z}$, this implies that $d = 4$ or $d = 8$.

Since $g : X \rightarrow Z$ is finite surjective, we obtain that $-K_Z$ is big by Lemma 5.4. Since Z_1 and Z_2 are smooth, $-K_Z$ is big if and only if $Z_i \rightarrow B_i$ is branched along a smooth conic for $i = 1, 2$ and $p = q = 1$. Hence $Z_1 \cong Z_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, Z is of degree $(1, 1, 1, 1)$ in $Z_1 \times Z_2 \cong (\mathbb{P}^1)^4$ and $K_X \equiv \frac{4}{d}g^*K_Z$. Hence by Lemma 5.6, we obtain that $X \cong Z$ and $d = 4$. Thus $a_1 = a_2 = 1$. \square

Lemma 5.11. *Let X be a Fano threefold with Gorenstein canonical singularities. Let $B_1 \cong B_2 \cong \mathbb{P}^2$. Assume that there are two equidimensional projective morphisms $f_1 : X \rightarrow B_1$ and $f_2 : X \rightarrow B_2$ such that general fibers of f_i have i connected components for $i = 1, 2$ and that $f_1 \times f_2 : X \rightarrow B_1 \times B_2$ is finite onto its image W . Assume that*

- W is a normal hypersurface with isolated \mathbb{Q} -factorial klt singularities of degree (p, q) , where $p, q \in \{1, 2\}$;
- If $X \rightarrow Z_2 \rightarrow B_2$ is the Stein factorisation of f_2 , then Z_2 is smooth;
- K_X is numerically equivalent to the pullback of some \mathbb{Q} -divisor in $B_1 \times B_2$ by $f_1 \times f_2$.

Then the degree of f is 2 and $q = 1$.

Proof. Let $Z = h^*W$, where h is the natural morphism $B_1 \times Z_2 \rightarrow B_1 \times B_2$. Then $Z \cong W \times_{B_2} Z_2$. Since W has isolated klt singularities, $W \rightarrow B_2$ has reduced fibers over some open subset of B_2 whose complement has codimension at least 2. By Lemma 3.3, $Z_2 \rightarrow B_2$ is a cyclic cover. Hence by Lemma 3.2, Z is normal. Thus Z equal to the image of X in $B_1 \times Z_2$. Moreover, since $-K_X$ is ample, $-K_Z$ is big by Lemma 5.4. Thus $q = 1$ and $Z_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of B_2 , branched along some smooth conic. Moreover, if $d = \deg f$, then $d \in 2\mathbb{Z}$.

Let $f : X \rightarrow W$ be the natural morphism. Let A_1, A_2 be two divisors in X such that $\mathcal{O}_X(A_1) \cong f_1^*\mathcal{O}_{B_1}(1)$ and $\mathcal{O}_X(A_2) \cong f_2^*\mathcal{O}_{B_2}(1)$. By assumption, there are two rational numbers a_1, a_2 such that $-K_X \equiv a_1A_1 + a_2A_2$. We have

$$A_1^2A_2 = qd = d \text{ and } A_1A_2^2 = pd.$$

Note that every component of a general fiber of f_i is a smooth rational curve for $i = 1, 2$. Hence we have

$$-K_X A_1^2 = 2 \text{ and } -K_X A_2^2 = 4.$$

This shows that

$$a_1 = \frac{4}{pd} \text{ and } a_2 = \frac{2}{d}.$$

We have

$$-K_X^3 = 3a_1a_2^2A_1A_2^2 + 3a_1^2a_2A_1^2A_2 = \frac{48}{d^2} \left(1 + \frac{2}{p^2} \right).$$

Since $-K_X^3$ is a positive even integer, we obtain that

$$r = \frac{24}{d^2} \left(1 + \frac{2}{p^2} \right)$$

is an integer. Since $d \in 2\mathbb{Z}$, this implies that $d = 2$ or 6 .

Assume that $d = 6$. Then $a_1 = \frac{2}{3p}$ and $a_2 = \frac{1}{3}$. The hypersurface Z is of degree $(p, 1, 1)$ in $B_1 \times Z_2 \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ and $X \rightarrow Z$ is of degree 3. Let φ, ψ be the natural projections from X to \mathbb{P}^1 induced by $X \rightarrow Z_2$. Let F and G be general fibers of φ and ψ respectively. Then $\mathcal{O}_X(F + G) \cong \mathcal{O}_X(A_2)$. We have

$$A_1^2F = 3, \quad A_1FG = 3p, \quad G^2F = 0$$

and

$$-K_X \equiv \frac{2}{3p}A_1 + \frac{1}{3}F + \frac{1}{3}G.$$

By the adjunction formula, we have

$$\begin{aligned} K_F^2 &= (K_X + F)^2 F = \left(\frac{2}{3p}A_1 + \frac{1}{3}G\right)^2 F \\ &= \frac{4}{9p^2}A_1^2 F + \frac{4}{9p}A_1 G F = \frac{4}{3p^2} + \frac{4}{3} = \frac{4}{3p^2}(p^2 + 1). \end{aligned}$$

Since F is a surface with canonical singularities, $K_F^2 \in \mathbb{Z}$. This implies that 3 divides $(p^2 + 1)$ which is impossible since $p \in \mathbb{Z}$.

Thus we have $d = 2$. \square

5.3. Proof of Theorem 1.2. Theorem 1.2 follows from Proposition 5.7 and Proposition 5.9.

Proof of Theorem 1.2. Let R_1, R_2 be the two extremal rays of the cone $\overline{\text{NE}}(X)$. Let $f_i : X \rightarrow B_i$ be the Mori fibration corresponding R_i for $i = 1, 2$. Then by Lemma 4.1, B_i is a Fano variety with Picard number 1 for $i = 1, 2$. By symmetry, we may assume that $\dim B_1 \leq \dim B_2$. Since $R_1 \cap R_2 = \{0\}$, the morphism $f_1 \times f_2 : X \rightarrow B_1 \times B_2$ is finite onto its image.

Assume first that $B_1 = \mathbb{P}^1$. Then B_2 is of dimension 2 since $f_1 \times f_2$ is finite. By Lemma 4.3, we have $B_2 \cong \mathbb{P}^2$. Since X has Picard number 2, by Proposition 5.7, X is one of the threefolds in Proposition 5.7. Since X is locally factorial, if f is of degree two, then f is branched along a prime divisor by [Dru14, Lem. 3.7]. Hence X is one of the threefolds of 1 – 3 of Theorem 1.2.

Assume that $\dim B_1 = 2$. Then $\dim B_2 = 2$. By Lemma 4.2, we have $B_1 \cong B_2 \cong \mathbb{P}^2$. Then X is one of the threefold in Proposition 5.9. Since X is locally factorial, as in the previous paragraph, we can conclude that X is one of the threefolds of 4 – 7 of Theorem 1.2. \square

The following observation will be useful for the proof of Theorem 1.5.

Lemma 5.12. *Let X be a threefold in 4 – 7 of Theorem 1.2. Let W be the image of the natural morphism $X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$. Then W is a normal variety with isolated locally factorial canonical singularities.*

Proof. If the natural morphism $X \rightarrow W$ is an isomorphism, then there is nothing to prove. If $X \rightarrow W$ is of degree 2, then W is smooth of degree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ by Proposition 5.9. \square

6. CASE OF DIMENSION 4

We will study Fano fourfold X with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = 3$ and $\text{Nef}(X) = \text{Psef}(X)$. Consider the cone $\overline{\text{NE}}(X)$. For every face V of dimension 2 of $\overline{\text{NE}}(X)$, there are exactly two faces V' and V'' , of dimension 2, whose intersections with V are extremal rays of $\overline{\text{NE}}(X)$. The aim of this section is to prove the following proposition.

Proposition 6.1. *If X is a Fano fourfold with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = 3$ and $\text{Nef}(X) = \text{Psef}(X)$, then there is a fibration from X to \mathbb{P}^1 .*

We will assume the existence of Fano fourfolds which satisfy the following condition $(*)$ and we will obtain a contradiction.

$(*)$ X is a Fano fourfold with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = 3$ and $\text{Nef}(X) = \text{Psef}(X)$. Moreover, there is no fibration from X to \mathbb{P}^1 .

We will first show that if X is a variety satisfying $(*)$, then it is a finite cover of the intersection of two hypersurfaces of degree $(0, 2, 2)$ and $(2, 2, 0)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Lemma 6.2. *Let X be a Fano fourfold satisfying the conditions in Proposition 6.1. Assume that there is no fibration from X to \mathbb{P}^1 . Let V_1, V_2, V_3 be three distinct faces of dimensions 2 in $\overline{\text{NE}}(X)$ such that $V_1 \cap V_2$ and $V_2 \cap V_3$ are extremal rays. Let f_i be the fibration from X to B_i corresponding to V_i ($i = 1, 2, 3$). Then $B_i = \mathbb{P}^2$ for all i and the morphism $f_1 \times f_2 \times f_3 : X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is finite onto its image Z . Moreover, Z is the intersection of hypersurfaces D and E of degree $(r, s, 0)$ and $(0, a, b)$ such that $r, s, a, b \in \{1, 2\}$.*

Proof. For all i , since the base B_i has Picard number 1 and there is no fibration from X to \mathbb{P}^1 , by Lemma 4.2, we obtain that $B_i \cong \mathbb{P}^2$.

Let $g : X \rightarrow Y$ be the Mori fibration corresponding to the extremal ray $R_{12} = V_1 \cap V_2$. Then g is equidimensional of relative dimension 1 by Lemma 4.8. By Lemma 4.2, Y is a Fano threefold with isolated

locally factorial canonical singularities such that $\rho(Y) = 2$ and $\text{Nef}(Y) = \text{Psef}(Y)$. Hence Y is one of the threefolds in 4 – 7 of Theorem 1.2. Let D' be the image of $f_1 \times f_2 : X \rightarrow B_1 \times B_2$. Since $f_1 \times f_2$ factorises through $g : X \rightarrow Y$, D' is a hypersurface of degree (r, s) in $B_1 \times B_2$ with $r, s \in \{1, 2\}$ by Theorem 1.2. By the Lefschetz theorem, the natural map $\text{Pic}(B_1 \times B_2) \rightarrow \text{Pic}(D')$ is an isomorphism (see [Laz04, Example 3.1.25]). Moreover, D' has isolated locally factorial canonical singularities by Lemma 5.12.

Let $h : X \rightarrow D'$ be the natural morphism. Since $R_{12} \cap V_3 = \{0\}$, the morphism $h \times f_3 : X \rightarrow D' \times B_3$ is finite onto its image Z . Let $D = D' \times B_3$. Then D is a hypersurface of degree $(r, s, 0)$ in $B_1 \times B_2 \times B_3$. Since B_3 is smooth, the product D is locally factorial. Hence Z is a Cartier divisor in D .

Note that the natural morphism $\text{Pic}(D') \times \text{Pic}(B_3) \rightarrow \text{Pic}(D)$ is an isomorphism since $B_3 = \mathbb{P}^2$ is a simply connected manifold (see [Har77, Ex. III.12.6]). Hence the natural morphism

$$\text{Pic}(B_1) \times \text{Pic}(B_2) \times \text{Pic}(B_3) \rightarrow \text{Pic}(D)$$

is an isomorphism. Since Z is a Cartier divisor in D , we obtain that there is a hypersurface E of degree (c, a, b) in $B_1 \times B_2 \times B_3$ such that Z is the intersection of E and D .

Since $V_3 \cap V_2 \neq \{0\}$, the fibers of the morphism $f_2 \times f_3 : X \rightarrow B_2 \times B_3$ have positive dimension by [Dru14, Lem. 4.4]. In particular, its image is a proper subvariety. Thus the image of $Z \rightarrow B_2 \times B_3$ is also a proper subvariety. This implies that $c = 0$.

Let $E' \subseteq B_2 \times B_3$ be the image of $f_2 \times f_3$. Since $f_2 \times f_3$ factors through $X \rightarrow Z$, we obtain that E' is of degree (a, b) in $B_2 \times B_3$. As in the second paragraph of the proof, we have $a, b \in \{1, 2\}$ and $E \cong B_1 \times E'$. This completes the proof of the lemma. \square

Lemma 6.3. *Let X be a Fano fourfold satisfying the conditions in Proposition 6.1. Assume that there is no fibration from X to \mathbb{P}^1 . Then the cone $\overline{\text{NE}}(X)$ is not simplicial.*

Proof. Assume the opposite. Since the cone $\overline{\text{NE}}(X)$ is simplicial, it has three faces V_1, V_2, V_3 of dimension 2 and three extremal rays $V_1 \cap V_2, V_2 \cap V_3$ and $V_3 \cap V_1$. As in the proof of Lemma 6.2, for every face V_i , there is a fibration $f_i : X \rightarrow \mathbb{P}^2$ corresponding V_i ($i = 1, 2, 3$). Let Z be the image of $f_1 \times f_2 \times f_3 : X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Then Z is the intersection of hypersurfaces of degree $(r, s, 0)$ and $(0, a, b)$ and that $r, s, a, b \in \{1, 2\}$. Since $V_3 \cap V_1 \neq \{0\}$, we obtain that the image of $f_1 \times f_3 : X \rightarrow B_1 \times B_3$ is a proper subvariety. Hence $as = 0$. This is a contradiction. \square

Lemma 6.4. *Let $W = B_1 \times B_2 \times B_3$, where $B_i = \mathbb{P}^2$ for $i = 1, 2, 3$. Let D' and E' be normal \mathbb{Q} -factorial klt hypersurfaces in $B_1 \times B_2$ and $B_2 \times B_3$ of degree $(a, 1)$ and (b, c) with $a, b, c \in \{1, 2\}$. Let $D = D' \times B_3$ and $E = B_1 \times E'$. In other words, D and E are hypersurfaces of degree $(a, 1, 0)$ and $(0, b, c)$ in W . Let Z be the intersection of D and E . If f_1, f_3 are the natural projections from Z to B_1, B_3 respectively, then $f = f_1 \times f_3 : Z \rightarrow B_1 \times B_3$ is not finite.*

Proof. Let p_1, p_2 (resp. q_2, q_3) be the natural projections from D' (resp. E') to B_1 and B_2 (resp. B_2 and B_3). Since D' and E' are \mathbb{Q} -factorial klt Fano varieties, the morphisms p_1, p_2, q_2, q_3 are equidimensional by Lemma 5.3. Thus $p_1 : D' \rightarrow B_1$ is a well-defined family of cycles in $B_2 = \mathbb{P}^2$ (see [Kol96, Def. I.3.11]). Since D' is of degree $(a, 1)$, the cycles in this family are lines in B_2 . Thus there exist a natural morphism ch from B_1 to the Grassmannian of lines in B_2 (see [Kol96, Thm. I.3.21]), which is isomorphic to \mathbb{P}^2 . Since $a \neq 0$, the morphism ch is not constant. Note that B_1 has Picard number 1. This shows that ch is finite. Hence it is surjective and for any line L in B_2 , there is a point x in B_1 such that the fiber of p_1 over x is mapped to L by the projection p_2 .

In order to prove the lemma, it is enough to find two points $x \in B_1$ and $z \in B_3$ such that $f^{-1}(\{(x, z)\})$ contains a curve. First we assume that $b = 1$. Let L be a line in B_2 . From the previous paragraph, there is a point $x \in B_1$ such that $p_1^{-1}(\{x\})$ is mapped to L by p_2 . Similarly, there is a point $z \in B_3$ such that $q_3^{-1}(\{z\})$ is mapped to L by q_2 . Hence the curve $\{(x, y, z) \in W \mid y \in L\}$ is contained in $f^{-1}(\{(x, z)\})$. Thus, f is not finite.

Now we assume that $b = 2$. Since q_3 is equidimensional, $q_3 : E' \rightarrow B_3$ is a well-defined family of cycles in $B_2 \cong \mathbb{P}^2$. Hence there is a morphism from B_3 to the Chow variety of conics in \mathbb{P}^2 . Since B_3 has Picard number 1 and $c \neq 0$, this morphism is finite onto its image. Note that there is a hyperplane H in the Chow variety of conics in \mathbb{P}^2 which parametrize the singular conics. Hence there is a point $z \in B_3$ such that the fiber of q_3 over z is the union of two lines L and L' . There is a point $x \in B_1$ such that $p_1^{-1}(\{x\})$ is mapped

to L by p_2 . This shows that the curve $\{(x, y, z) \in W \mid y \in L\}$ is contained in $f^{-1}(\{(x, z)\})$. Hence f is not finite. \square

Lemma 6.5. *With the notation in Lemma 6.2, the numbers a, b, r, s are all equal to 2. Moreover, if we write $D = D' \times B_3$ and $E = B_1 \times E'$, then the natural projections from X to D' and E' are equidimensional Mori fibrations.*

Proof. By Lemma 6.3, the cone $\overline{NE}(X)$ is not simplicial. Hence $V_1 \cap V_3 = \{0\}$. This implies that $f_1 \times f_3 : X \rightarrow B_1 \times B_3$ is finite surjective. Since $f_1 \times f_3$ factors through $X \rightarrow Z$, the natural morphism $Z \rightarrow B_1 \times B_2$ is finite surjective. By Lemma 5.12, D' and E' are Fano varieties with isolated locally factorial canonical singularities. Thus by Lemma 6.4, we have $s = a = 2$.

Note that there is a face V_4 of dimension 2 of $\overline{NE}(X)$ such that $V_2 \neq V_4$ and $V_3 \cap V_4$ is an extremal ray. Let $f_4 : X \rightarrow B_4$ be the fibration corresponding to V_4 . Then, as in Lemma 6.2, $B_4 = \mathbb{P}^2$ and

$$f_2 \times f_3 \times f_4 : X \rightarrow B_2 \times B_3 \times B_4$$

is finite onto its image T . Moreover, T is the intersection of hypersurfaces of degree of $(a, b, 0)$ and $(0, p, q)$ such that $p, q \in \{1, 2\}$. As in the previous paragraph, we have $b = p = 2$. Similarly, we can obtain that $r = 2$.

As in the proof of Lemma 6.2, D' is in fact equal to the image of $f_1 \times f_2 : X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$. Let $g : X \rightarrow Y$ be the Mori fibration corresponding the extremal ray $V_1 \cap V_2$. Then $X \rightarrow Y \rightarrow D'$ is just the Stein factorisation of $X \rightarrow D'$. Since D' is of degree $(2, 2)$ and Y is one of the varieties in 4–7 of Theorem 1.2, we obtain that $Y \rightarrow D'$ is an isomorphism. Hence $X \rightarrow D'$ is an equidimensional Mori fibration by Lemma 4.8. Similarly, $X \rightarrow E'$ is also an equidimensional Mori fibration. \square

Lemma 6.6. *With the notation in Lemma 6.2, the image Z of $f_1 \times f_2 \times f_3 : X \rightarrow B_1 \times B_2 \times B_3$ is normal.*

Proof. By Lemma 6.5, we have $Z = D \cap E$, where $D \cong D' \times B_3$ and $E \cong B_1 \times E'$ are hypersurfaces of degree $(2, 2, 0)$ and $(0, 2, 2)$ respectively in $B_1 \times B_2 \times B_3$.

By Lemma 5.12, E' is a Fano threefold with isolated singularities. Hence, by Lemma 5.1, there is an open subset U of B_2 whose complement is of codimension at least 2 such that the natural projection $q_2 : E' \rightarrow B_2$ has reduced fibers over U .

Consider the natural projection $\varphi : Z \rightarrow D'$. Let α be a point in D' . Then the fiber of φ over α is a conic in B_3 . Moreover, if β is the image of α by the natural projection $p_2 : D' \rightarrow B_2$, then the fiber of q_2 over β is isomorphic to the fiber of φ over α . Hence general fibers of φ are smooth conics in B_3 and the fibers of φ over $p_2^{-1}(U) \subseteq D'$ are reduced.

Note that D' is a Fano threefold with locally factorial canonical singularities by Lemma 5.12. Thus $p_2 : D' \rightarrow B_2$ is equidimensional by Lemma 5.3. Since $\text{codim } B_2 \setminus U \geq 2$, the complement of $p_2^{-1}(U)$ in D' has codimension at least 2. Let

$$V = \{z \in Z \mid \varphi^{-1}(\{\varphi(z)\}) \text{ is singular at } z\}.$$

Then $\text{codim } V \geq 2$. Since Z is Cohen-Macaulay and φ is equidimensional, φ is flat over the smooth locus of D' . Since D' is smooth in codimension 1, this implies that Z is smooth in codimension 1. Thus Z is normal for it is Cohen-Macaulay. \square

Now we can prove Proposition 6.1.

Proof of Proposition 6.1. Assume the opposite. With the same notation as in Lemma 6.2, there are fibrations $f_i : X \rightarrow B_i$ such that $f_1 \times f_2 \times f_3 : X \rightarrow B_1 \times B_2 \times B_3$ is finite onto its image Z . By Lemma 6.5, Z is the intersection of $D' \times B_3$ and $B_1 \times E'$ such that D' is normal of degree $(2, 2)$ in $B_1 \times B_2$ and E' is normal of degree $(2, 2)$ in $B_2 \times B_3$. Moreover, $g : X \rightarrow D'$ is an equidimensional Mori fibration.

By Lemma 6.6, Z is normal. Let $p_i : Z \rightarrow B_i$ be the natural fibration and let H_i be a divisor in Z such that $\mathcal{O}_Z(H_i) \cong p_i^* \mathcal{O}_{B_i}(1)$ for $i = 1, 2, 3$. Then

$$-K_Z \equiv H_1 - H_2 + H_3.$$

Since X is a Fano variety, $-K_Z$ is big by Lemma 5.4. Moreover, since $\overline{NE}(X)$ is not simplicial by Lemma 6.3, as in the proof of Lemma 6.5, $f_1 \times f_3 : X \rightarrow B_1 \times B_3$ is finite. We obtain that $p_1 \times p_3 : Z \rightarrow B_1 \times B_3$

is also finite. Hence $H_1 + H_3$ is an ample divisor and the intersection number $-K_Z(H_1 + H_3)^3$ is positive. However, we have

$$\begin{aligned} -K_Z(H_1 + H_3)^3 &= (H_1 + H_3 - H_2)(3H_1^2H_3 + 3H_1H_3^2) \\ &= 6H_1^2H_3^2 - 3H_1^2H_2H_3 - 3H_1H_2H_3^2 = 0. \end{aligned}$$

We obtain a contradiction. \square

7. FINITE MORPHISMS BETWEEN FANO VARIETIES.

In this section, we will study some finite morphisms between Fano varieties.

7.1. Finite morphisms over $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$.

Lemma 7.1. *Let Y be one of the varieties in Example 1.3.2. Set $n = \dim Y$. Assume that Y satisfies the conditions of Theorem 1.5. Let X be a Fano variety with Gorenstein canonical singularities such that there is a finite surjective morphism $f : X \rightarrow Y$. Assume that the projection from X to $(\mathbb{P}^1)^{n-2}$ induced by f has connected fibers. Then f is an isomorphism.*

Proof. From the construction of Y , we have two double covers

$$Y \xrightarrow{\pi_1} T \xrightarrow{\pi_2} (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2,$$

where π_2 is branched along some prime divisor of degree $(2, \dots, 2, 0, \dots, 0, 2)$. In particular, $T \cong (\mathbb{P}^1)^r \times T'$ where T' is one of the varieties in Example 1.3.1. Note that $Y \rightarrow T' \rightarrow (\mathbb{P}^1)^{n-2-r} \times \mathbb{P}^2$ is the Stein factorisation. By Lemma 4.3, we obtain that T' is smooth in codimension 2. Thus T is smooth in codimension 2.

We write

$$(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2 = (\mathbb{P}^1)^{n-3} \times (\mathbb{P}^1 \times \mathbb{P}^2) = (\mathbb{P}^1)^{n-3} \times Z.$$

Let p be the natural morphism from Y to $(\mathbb{P}^1)^{n-3}$ and let h be the natural morphism from Y to Z . If F is the fiber of p over a general point s , then F is connected and $h|_F : F \rightarrow Z$ is the composition of two double covers

$$F \xrightarrow{\pi_1|_F} V \xrightarrow{\pi_2|_V} \{s\} \times Z,$$

where $V \cong \mathbb{P}^1 \times V'$ such that V' is a general fiber of $T \rightarrow (\mathbb{P}^1)^{n-2}$. Since T is smooth in codimension 2, V' is smooth. Thus $V' \rightarrow \mathbb{P}^2$ is a double cover branched along some smooth conic by Lemma 3.3.

Hence $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Assume that $F \rightarrow V$ is branched along a divisor D . Then D is the pullback of some divisor of degree $(2, 2)$ in Z by construction. Thus D is a divisor of degree $(2, 2, 2)$ in $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let G be the fiber of the natural morphism $X \rightarrow (\mathbb{P}^1)^{n-3}$ over s . Then $f|_G : G \rightarrow F$ is also a finite surjective morphism. Since s is a general point, G is a Fano variety with Gorenstein canonical singularities. By [Dru14, Thm. 1.3], $f|_G$ is of degree 1. Hence f is of degree 1 and is an isomorphism. \square

Lemma 7.2. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. Set $n = \dim X$. Assume that Theorem 1.5 is true in dimension smaller than n . Assume that there is a fibration $X \rightarrow (\mathbb{P}^1)^{n-2}$ and a fibration $X \rightarrow \mathbb{P}^2$ such that the product morphism $f : X \rightarrow (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ is finite surjective. Then X is one of the varieties in Theorem 1.5.*

Proof. Note that if $X \rightarrow Y$ is the Stein factorisation of some projection

$$X \rightarrow (\mathbb{P}^1)^{n-3} \times \mathbb{P}^2 = Z \times \mathbb{P}^2,$$

then Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\text{Nef}(Y) = \text{Psef}(Y)$ by Lemma 4.3. Moreover, since $X \rightarrow (\mathbb{P}^1)^{n-2}$ has connected fibers, by Theorem 1.5 in dimension $n - 1$, Y is of the form $(\mathbb{P}^1)^{s-1} \times Y_1$, where $s \geq 1$ and Y_1 is one of the varieties in Example 1.3 and Theorem 1.2. Thus $Y \rightarrow Z \times \mathbb{P}^2$ is of degree 1, 2 or 4. Moreover, by Lemma 4.3, Y_1 is a Fano variety with locally factorial canonical singularities such that Y_1 is smooth in codimension 2, $\rho(Y_1) = \dim Y_1 - 1$ and $\text{Nef}(Y_1) = \text{Psef}(Y_1)$. Let $f_1 : X \rightarrow \mathbb{P}^1$ be the projection to the remaining factor.

Let A_1 be a fiber of f_1 . Let A_2, \dots, A_{n-2} be the fibers of X over the other \mathbb{P}^1 . Let H be the pullback of a line in \mathbb{P}^2 by the natural projection $X \rightarrow \mathbb{P}^2$. Note that K_X is numerically equivalent to the pullback of

some \mathbb{Q} -divisor in $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ since $\rho(X) = n - 1$. Hence there are positive rational numbers a_1, \dots, a_{n-2}, b such that $-K_X \equiv a_1 A_1 + \dots + a_{n-2} A_{n-2} + bH$.

We will discuss on three case.

Case 1. Assume that the morphism $Y \rightarrow Z \times \mathbb{P}^2$ is of degree 4. Then Y_1 is one of the varieties in Example 1.3.2 and X is finite over $(\mathbb{P}^1)^s \times Y_1$. The natural morphism $X \rightarrow (\mathbb{P}^1)^s$ is a fibration since $X \rightarrow (\mathbb{P}^1)^{n-2}$ is. Let F be a general fiber of $X \rightarrow (\mathbb{P}^1)^s$. Then F is a Fano variety with Gorenstein canonical singularities. Since $X \rightarrow (\mathbb{P}^1)^{n-2}$ has connected fibers, the natural morphism $F \rightarrow (\mathbb{P}^1)^{n-2-s}$ given by $X \rightarrow Y_1 \rightarrow (\mathbb{P}^1)^{n-2-s}$ is a fibration. By Lemma 7.1, the natural finite morphism $F \rightarrow Y_1$ is an isomorphism. Hence $X \cong (\mathbb{P}^1)^s \times Y_1$ in this case.

Case 2. Assume that the morphism $Y \rightarrow Z \times \mathbb{P}^2$ is of degree 2. Let G be a general fiber of the natural projection $X \rightarrow Z$. Then G is a Fano threefold with Gorenstein canonical singularities. Moreover, the restriction $f_1|_G : G \rightarrow \mathbb{P}^1$ is a fibration, general fibers of the natural morphism $G \rightarrow \mathbb{P}^2$ have two connected components and the product $h : G \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is finite. Note that $K_G \equiv a_1 A_1|_G + bH|_G$. By Lemma 5.8, h is of degree 2 or 4 and $b \in \mathbb{Z}$. Hence f is of degree 2 or 4.

The Cartier divisor $-K_X - bH$ is numerically trivial on the fibers of the Fano fibration $X \rightarrow (\mathbb{P}^1)^{n-2}$. Thus it is linearly equivalent to the pullback of some Cartier divisor in $(\mathbb{P}^1)^{n-2}$ by [KMM87, Lem. 3-2-5]. This implies that $a_1, \dots, a_{n-2} \in \mathbb{Z}$.

Note that Y_1 is a double cover of $(\mathbb{P}^1)^r \times \mathbb{P}^2$, branched along some prime divisor of degree $(2, \dots, 2, k)$ such that $k \in \{2, 4\}$ and $r + s = n - 2$. If $\deg f = 2$, then $X \cong (\mathbb{P}^1)^s \times Y_1$. If $\deg f = 4$, then X is a double cover of $V = \mathbb{P}^1 \times Y = (\mathbb{P}^1)^s \times Y_1$.

There is a natural double cover $h : V \rightarrow (\mathbb{P}^1)^s \times (\mathbb{P}^1)^r \times \mathbb{P}^2$, induced by $Y_1 \rightarrow (\mathbb{P}^1)^r \times \mathbb{P}^2$, branched along a prime divisor of degree $(0, \dots, 0, 2, \dots, 2, k)$. Since $a_1, \dots, a_{n-2}, b \in \mathbb{Z}$ and $\rho(V) = n - 1$, the double cover $X \rightarrow V$ is branched along a divisor $D \subseteq V$ which is numerically equivalent to the pullback of a divisor of degree $(d_1, \dots, d_s, 0, \dots, 0, e)$ in $(\mathbb{P}^1)^s \times (\mathbb{P}^1)^r \times \mathbb{P}^2$ such that $d_1, \dots, d_s, e \in \{0, 2\}$. Since $X \rightarrow (\mathbb{P}^1)^{n-2}$ has connected fibers, we obtain that $e > 0$. Hence $e = 2$. Since $b > 0$, we obtain that $k = 2$. Moreover, since $X \rightarrow \mathbb{P}^2$ has connected fibers, d_1, \dots, d_s are not all zero. Note that V is simply connected and locally factorial. Thus D is in fact linearly equivalent to the pullback of a divisor of degree $(d_1, \dots, d_s, 0, \dots, 0, e)$ in $(\mathbb{P}^1)^s \times \mathbb{P}^2 \times (\mathbb{P}^1)^r$. Note that D is a prime divisor by [Dru14, Lem. 3.7] Thus we obtain that $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^t$ with $s > t \geq 0$ and X_2 is one of the varieties in Example 1.3.2.

Case 3. Assume $Y \rightarrow Z \times \mathbb{P}^2$ is an isomorphism. Let G be a general fiber of the natural projection $X \rightarrow Z$. Then G is a Fano threefold with Gorenstein canonical singularities. Moreover, $f_1|_G : G \rightarrow \mathbb{P}^1$ and the natural morphism $G \rightarrow \mathbb{P}^2$ are fibrations, and the product $h : G \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is finite. By Proposition 5.7, h is of degree at most 2. Hence f is of degree at most 2.

If the degree of f is 1, then f is an isomorphism. If $\deg f = 2$, then by Lemma 3.3 and Lemma 3.4, $f : X \rightarrow (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ is a double cover, branched along a prime divisor of degree (d_1, \dots, d_{n-2}, k) such that $d_1, \dots, d_{n-2} \in \{0, 2\}$ and that $k \in \{0, 2, 4\}$. Since $X \rightarrow (\mathbb{P}^1)^{n-2}$ and $X \rightarrow \mathbb{P}^2$ have connected fibers, we obtain that $k \neq 0$ and d_1, \dots, d_{n-2} are not all zero. Hence $X \cong X_1 \times X_2$, where $X_1 \cong (\mathbb{P}^1)^t$ for some $t \geq 0$ and X_2 is a double cover of $(\mathbb{P}^1)^{n-2-t} \times \mathbb{P}^2$, branched along an prime divisor of degree $(2, \dots, 2, 2)$ or $(2, \dots, 2, 4)$. \square

7.2. Finite morphisms over hypersurfaces of $(\mathbb{P}^1)^{n-3} \times \mathbb{P}^2 \times \mathbb{P}^2$.

Lemma 7.3. *Let Y be one of the varieties in Example 1.3.2 of dimension n . Then there is a finite surjective morphism, of degree 4, $Y \rightarrow (\mathbb{P}^1)^{n-2} \times B_1$ with $B_1 \cong \mathbb{P}^2$. Let X be a hypersurface in $Y \times B_2$, where $B_2 \cong \mathbb{P}^2$. Assume that the image of X in $B_1 \times B_2$ by the natural projection is a normal hypersurface W of degree (p, q) with isolated \mathbb{Q} -factorial klt singularities, where $p, q \in \{1, 2\}$. Then X is normal and $-K_X$ is not big.*

Proof. There is a natural finite surjective morphism $X \rightarrow D$, where

$$D = (\mathbb{P}^1)^{n-2} \times W \subseteq (\mathbb{P}^1)^{n-2} \times B_1 \times B_2$$

is a normal hypersurface. By the construction of Y , the natural morphism $Y \times B_2 \rightarrow (\mathbb{P}^1)^{n-2} \times B_1 \times B_2$ is the composition of the following two double covers

$$Y \times B_2 \xrightarrow{g} V \times B_2 \xrightarrow{h} (\mathbb{P}^1)^{n-2} \times B_1 \times B_2.$$

Note that the natural morphism $D \rightarrow (\mathbb{P}^1)^{n-2} \times B_1$ is an equidimensional Fano fibration since $W \rightarrow B_1$ is equidimensional by Lemma 5.3. Since W has isolated singularities, D is smooth in codimension 2. There is an open subset U in $(\mathbb{P}^1)^{n-2} \times B_1$ whose complement has codimension at least 2 such that $D \rightarrow (\mathbb{P}^1)^{n-2} \times B_1$

has reduced fibers over U by Lemma 5.1. Note that $V \rightarrow (\mathbb{P}^1)^{n-2} \times B_1$ is a cyclic cover branched along some prime divisor and that

$$h^*D \cong D \times_{(\mathbb{P}^1)^{n-2} \times B_1} V.$$

Hence, h^*D is normal Cohen-Macaulay by Lemma 3.2. Moreover, $h^*D \rightarrow V$ also has reduced fibers over some open subset of V whose complement has codimension at least 2. Similarly, by Lemma 3.2 again,

$$g^*(h^*D) = (h^*D) \times_V Y$$

is also normal. Hence, $X = g^*(h^*D)$ and it is normal.

Let A_1, \dots, A_{n-2} be the fibers of the natural projections from X to the factors \mathbb{P}^1 . Let H_i be a divisor in X which is the pullback of a line in B_i for $i = 1, 2$. Then by the adjunction formula, we have

$$-K_X \equiv A_1 + \dots + A_{n-2} + (1-p)H_1 + (3-q)H_2.$$

Since $p \geq 1$, $-K_X$ is not big. □

Lemma 7.4. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1$ and $\text{Nef}(X) = \text{Psef}(X)$. Set $n = \dim X$. Assume that Theorem 1.5 is true in dimension smaller than n . Assume that there are fibrations $X \rightarrow (\mathbb{P}^1)^{n-3}$, $X \rightarrow B_1$ and $X \rightarrow B_2$ such that $B_1 \cong B_2 \cong \mathbb{P}^2$ and that the product morphism $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$ is finite onto its image. Then X is one of the varieties in Theorem 1.5*

Proof. Let W be the image of $X \rightarrow B_1 \times B_2$ and let $X \rightarrow W' \rightarrow W$ be the Stein factorisation. Since the natural morphism $X \rightarrow (\mathbb{P}^1)^{n-3} \times W'$ is finite surjective, by Lemma 4.3, W' is a Fano threefold of Theorem 1.2. Thus by Lemma 5.12, W is a normal hypersurface of degree (p, q) , with isolated locally factorial canonical singularities, such that $p, q \in \{1, 2\}$. Let $f : X \rightarrow (\mathbb{P}^1)^{n-3} \times W$ be the natural morphism.

Let A_1, \dots, A_{n-3} be the fibers of the natural fibrations from X to \mathbb{P}^1 . Let H_i be the pullback of a line in B_i by the fibration from X to B_i for $i = 1, 2$. Since the Picard number of X is $n - 1$, there are positive rational numbers $a_1, \dots, a_{n-3}, b_1, b_2$ such that $-K_X \equiv a_1 A_1 + \dots + a_{n-3} A_{n-3} + b_1 H_1 + b_2 H_2$. Let F be the fiber of $X \rightarrow (\mathbb{P}^1)^{n-3}$ over a general point $\alpha \in (\mathbb{P}^1)^{n-3}$, then $-K_F \equiv (b_1 H_1 + b_2 H_2)|_F$. Moreover, F is a Fano threefold with isolated Gorenstein canonical singularities and the natural morphism $F \rightarrow W$ is finite.

Let $X \rightarrow Y$ be the Stein factorisation of $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$. Then $X \rightarrow Y$ is a Mori fibration of relative dimension 1 (see Lemma 4.4). By Lemma 4.1, Y is a Fano variety with locally factorial canonical singularities such that $\rho(Y) = \dim Y - 1$ and $\text{Nef}(Y) = \text{Psef}(Y)$. The fibration $X \rightarrow Y$ is equidimensional since $(\mathbb{P}^1)^{n-3} \times W \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$ is by Lemma 5.3. Hence by Lemma 4.5, Y is smooth in codimension 2. Thus, by induction, Y is one of the varieties in Theorem 1.5. In particular, $Y \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$ is of degree 1, 2 or 4.

If it is of degree 4, then $Y = (\mathbb{P}^1)^r \times X'$ where $r \geq 0$ and X' is one of the varieties in Example 1.3.2. Let S be a general fiber of $X \rightarrow (\mathbb{P}^1)^r$. Then S is a Fano variety. If T is the image of S in $X' \times B_2$, then by Lemma 7.3, T is normal and $-K_T$ is not big. Thus S can not be a Fano variety by Lemma 5.4. This is a contradiction. Hence the degree of $Y \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$ is 1 or 2. Similarly, fibers of $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_2$ have at most two connected components.

We will discuss on four cases.

Case 1. Assume that the projection $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_i$ is a fibration for $i = 1, 2$. Then the natural projections from F to B_1 and B_2 are fibrations. By Proposition 5.9, the finite morphism $F \rightarrow W$ is of degree 1 or 2.

If this degree is 1, then $f : X \rightarrow (\mathbb{P}^1)^{n-3} \times W$ is an isomorphism. If the degree of $F \rightarrow W$ is 2, then f is a double cover. Moreover, W is smooth of degree $(1, 1)$ in $B_1 \times B_2$ by Proposition 5.9. Note that there is a natural isomorphism $\text{Pic}((\mathbb{P}^1)^{n-3} \times B_1 \times B_2) \cong \text{Pic}((\mathbb{P}^1)^{n-3} \times W)$. By Lemma 3.3 and Lemma 3.4, f is branched along a prime divisor D which is the intersection of $(\mathbb{P}^1)^{n-3} \times W$ and a divisor of degree $(d_1, \dots, d_{n-3}, k, l)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$ such that $d_1, \dots, d_{n-3}, k, l \in \{0, 2\}$. Since the two morphisms $F \rightarrow B_1$ and $f \rightarrow B_2$ are fibrations, $k, l \neq 0$. Hence $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^r$ with some $r \geq 0$ and X_2 is one of the varieties in Example 1.4.1.

Case 2. Assume that $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_2$ is a fibration and general fibers of $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$ have two connected components. Then $Y = (\mathbb{P}^1)^r \times Y_1$, where $r \geq 0$ and Y_1 is one of the varieties in Example 1.3.1, Theorem 1.2.2 and Theorem 1.2.3. If $F \rightarrow Z_1 \rightarrow B_1$ is the Stein factorisation, then Z_1 is isomorphic to the fiber of $Y \rightarrow (\mathbb{P}^1)^{n-3}$ over α . Since Y is smooth in codimension 2, Z_1 is smooth. Thus the F satisfies the

conditions in Lemma 5.11. This implies that $p = 1$ and $F \rightarrow W$ is of degree 2. Thus $X \rightarrow (\mathbb{P}^1)^{n-3} \times W$ is of degree 2

Let V be the pullback of $(\mathbb{P}^1)^{n-3} \times W$ by the natural morphism $Y \times B_2 \rightarrow (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. As in the proof of Lemma 7.3, V is normal. Since $X \rightarrow (\mathbb{P}^1)^{n-3} \times W$ is of degree 2, we obtain that $X \cong V$. Since X is a Fano variety, we obtain that the double cover $Y_1 \rightarrow (\mathbb{P}^1)^{n-3-r} \times B_1$ is branched along some divisor of degree $(2, \dots, 2, 2)$. Hence $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^r$ and X_2 is one of the varieties in Example 1.4.2.

Case. 3. Assume that $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$ is a fibration and general fibers of $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_2$ have two connected components. By symmetry, we can reduce to the second case.

Case 4. Assume that general fibers of the two projections $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_1$ and $X \rightarrow (\mathbb{P}^1)^{n-3} \times B_2$ have two connected components. As in the second case, if $F \rightarrow Z_i \rightarrow B_i$ is the Stein factorisation for $i = 1, 2$, then Z_1 and Z_2 are smooth. Hence F satisfies the conditions in Lemma 5.10. The morphism $F \rightarrow W$ is of degree 4 and $p = q = b_1 = b_2 = 1$. Hence the morphism $X \rightarrow (\mathbb{P}^1)^{n-3} \times W$ is of degree 4. Moreover, $-K_X - b_1 H_1 - b_2 H_2$ is numerically trivial on the fibers of the Fano fibration $X \rightarrow (\mathbb{P}^1)^{n-3}$. Thus we obtain that $a_1, \dots, a_{n-3} \in \mathbb{Z}$.

We have $Y = (\mathbb{P}^1)^r \times Y_1$, where $r \geq 0$ and Y_1 is one of the varieties in Example 1.3.1, Theorem 1.2.2 and Theorem 1.2.3. Let V be the pullback of $(\mathbb{P}^1)^{n-3} \times W$ by the natural morphism $Y \times B_2 \rightarrow (\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Then V is a normal divisor by the same argument in the proof of Lemma 7.3 and $X \rightarrow V$ is a double cover. Note that here is an open subset U of $(\mathbb{P}^1)^{n-3} \times B_2$ whose complement has codimension at least 2 such that the $(\mathbb{P}^1)^{n-3} \times W \rightarrow (\mathbb{P}^1)^{n-3} \times B_2$ has reduced fibers over U , for W is smooth in codimension 2 (see Lemma 5.1). The branch locus of $V \rightarrow (\mathbb{P}^1)^{n-3} \times W$ is the intersection of $(\mathbb{P}^1)^{n-3} \times W$ and a divisor of degree $(0, \dots, 0, 2, \dots, 2, 2, 0)$ in $(\mathbb{P}^1)^{n-3} \times B_1 \times B_2$. Thus this branch locus is horizontal over $(\mathbb{P}^1)^{n-3} \times B_2$. Hence, the natural morphism $V \rightarrow (\mathbb{P}^1)^{n-3} \times B_2$ is a fibration and it has reduced fibers over some open subset $U' \subseteq U$ such that $\text{codim } U \setminus U' \geq 2$. Thus the pullback of any prime divisor in $(\mathbb{P}^1)^r \times B_2$ by the fibration $V \rightarrow (\mathbb{P}^1)^r \times B_2$ is reduced.

Assume that $X \rightarrow V$ is branched along some divisor D . Then D is a prime divisor by [Dru14, Lem. 3.7] for X is locally factorial. Since $a_1, \dots, a_{n-3} \in \mathbb{Z}$ and $b_1 = b_2 = 1$, we obtain that $Y_1 \rightarrow (\mathbb{P}^1)^{n-3-r} \times B_1$ is branched along some divisor of degree $(2, \dots, 2, 2)$ and D is numerically equivalent to the pullback of a divisor E of degree $(d_1, \dots, d_r, 2)$ in $(\mathbb{P}^1)^r \times B_2$ by the natural projection $V \rightarrow (\mathbb{P}^1)^r \times B_2$ such that $d_1, \dots, d_r \in \{0, 2\}$. Since the pullback of any prime divisor in $(\mathbb{P}^1)^r \times B_2$ by the fibration $V \rightarrow (\mathbb{P}^1)^r \times B_2$ is reduced, we have $D = f^*(f(D))$ and D is in fact linearly equivalent to the pullback of E .

Since the projection from X to B_2 is a fibration, we obtain that $r \geq 1$ and that d_1, \dots, d_r are not all equal to 0. Hence we obtain that $X \cong X_1 \times X_2$, where $X_1 = (\mathbb{P}^1)^t$ for some $r > t \geq 0$ and X_2 is one of the varieties in Example 1.4.3. \square

8. PROOF OF THEOREM 1.5

We will first prove the following lemma.

Lemma 8.1. *Let X be a Fano variety with locally factorial canonical singularities such that X is smooth in codimension 2, $\rho(X) = \dim X - 1 \geq 3$ and $\text{Nef}(X) = \text{Psef}(X)$. Then there is a fibration from X to \mathbb{P}^1 .*

Proof. We will argue by contradiction. Set $n = \dim X$. Assume that there is no fibration from X to \mathbb{P}^1 . Let V be a face of $\overline{\text{NE}}(X)$ of dimension $n - 4$. Then there is a morphism $X \rightarrow Y$ corresponding to V . By Lemma 4.2, Y is a Fano variety with locally factorial canonical singularities such that Y is smooth in codimension 2, $\rho(Y) = \dim Y - 1$ and $\text{Nef}(Y) = \text{Psef}(Y)$. Note that $\rho(Y) = (n - 1) - (n - 4) = 3$. Hence $\dim Y = 4$. By Proposition 6.1, there is a fibration from Y to \mathbb{P}^1 which can induce a fibration from X to \mathbb{P}^1 . This is a contradiction. \square

Now we can prove Theorem 1.5.

Proof of Theorem 1.5. We will prove by induction on the dimension of X . If $\dim X = 3$, then the theorem follows from Theorem 1.2. Assume that the theorem is true in dimension smaller than n , where $n \geq 4$ is an integer. Now we consider the case of $\dim X = n$.

By Lemma 8.1, there is a fibration $f_1 : X \rightarrow \mathbb{P}^1$. There is an extremal ray R in $\overline{\text{NE}}(X)$ such that the class of any curve in X contracted by f_1 is not contained in R . Let $f_2 : X \rightarrow Y$ be the fibration corresponding to R . Then $f = f_1 \times f_2 : X \rightarrow \mathbb{P}^1 \times Y$ is a finite surjective morphism. By assumption, Theorem 1.5 is true

in dimension smaller than n . Hence Y is one of the varieties in Theorem 1.5 by Lemma 4.3. We have the following two possibilities.

Case 1. The variety Y is a finite cover of $(\mathbb{P}^1)^{n-3} \times \mathbb{P}^2$ and f induces a finite surjective morphism $g : X \rightarrow (\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ such that the projections from X to \mathbb{P}^1 and the projection g_2 from X to \mathbb{P}^2 induced by g are fibrations. Let $X \xrightarrow{g_1} Z \xrightarrow{\pi} (\mathbb{P}^1)^{n-2}$ be the Stein factorisation of the natural projection induced by g . Then Z is one of the varieties in Theorem 1.1 by Lemma 4.1 and Lemma 4.6.

If π is of degree at least 2, then we can write $Z \cong Z_1 \times Z_2$ such that Z_1 is a double cover of $(\mathbb{P}^1)^r$ for some $r \geq 3$, branched along some prime divisor of degree $(2, \dots, 2)$, and Z_2 is one of the varieties of Theorem 1.1, of dimension $n - 2 - r$. Let $h_1 : X \rightarrow Z_1$ be the natural fibration. Let $h_2 : X \rightarrow X'$ be the Stein factorisation of the natural projection $X \rightarrow Z_2 \times \mathbb{P}^2$. Then, by Lemma 4.3, X' is a Fano variety with locally factorial canonical singularities such that X' is smooth in codimension 2, $\rho(X') = \dim X' - 1$ and $\text{Nef}(X') = \text{Psef}(X')$. Hence it is one of the varieties in Theorem 1.5 by induction hypotheses. Let F be a general fiber of h_2 . Then $h_1|_F : F \rightarrow Z_1$ is a finite surjective morphism. Since F is a Fano variety with canonical Gorenstein singularities, h_1 is an isomorphism by [Dru14, Thm. 1.3]. Hence $X \cong Z_1 \times X'$ and it is one of the varieties in Theorem 1.5.

If π is of degree 1, then $Z = (\mathbb{P}^1)^{n-2}$. By Lemma 7.2, X is one of the varieties in Theorem 1.5.

Case 2. The variety Y is a finite cover of $(\mathbb{P}^1)^{n-4} \times W$, where W is a normal \mathbb{Q} -factorial divisor of degree (p, q) in $B_1 \times B_2 \cong \mathbb{P}^2 \times \mathbb{P}^2$ such that $p, q \in \{1, 2\}$. Moreover, f induces a finite surjective morphism $g : X \rightarrow (\mathbb{P}^1)^{n-3} \times W$ such that the projections from X to \mathbb{P}^1 and the projections from X to B_1 and B_2 induced by g are fibrations. Let $X \xrightarrow{g_1} Z \xrightarrow{\pi} (\mathbb{P}^1)^{n-3}$ be the Stein factorisation of the natural projection induced by g . As in the first case, if π is of degree at least 2, then $X \cong Z_1 \times X'$, where Z_1 is a double cover of $(\mathbb{P}^1)^r$ for some $r \geq 3$, branched along some prime divisor of degree $(2, \dots, 2)$, and

$$X \rightarrow X' \rightarrow (\mathbb{P}^1)^{n-3-r} \times W$$

is the Stein factorisation. By Lemma 4.3 and induction hypotheses, X' is one of the varieties in Theorem 1.5.

If π is of degree 1, then $Z = (\mathbb{P}^1)^{n-3}$. By Lemma 7.4, X is one of the varieties in Theorem 1.5. □

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